



LECTURE 5

Waves and Oscillations

From Basic Concepts to Traveling and Standing Waves

Oscillations • Wave Properties • Traveling Waves • Standing Waves
Boundary Conditions • Applications



Understand



Model



Analyze



Apply



Engineer

"Waves are nature's way of transferring energy and information."

Why Waves Matter



Waves appear everywhere in engineering and nature. They help us **transmit energy, carry information**, and reveal the properties of materials and systems.

Waves in the Real World – Key Applications

1 Sound and Acoustics



- Pressure disturbances travel through air as sound waves.
- Loudspeakers, microphones, acoustic sensors, noise control, room acoustics.

2 Structural Vibrations



- Bridges, buildings, aircraft wings, machines and rotating equipment all have natural frequencies.
- If excitation matches a natural frequency, resonance can cause large amplitudes.

3 Seismic Waves



- Seismic waves travel through the Earth.
- They carry information about earthquakes and about underground structures.

4 Medical Technology



- Ultrasound uses acoustic waves in biological tissue.
- Reflection at interfaces forms images for diagnostics and monitoring.

5 Electromagnetic Waves



- Radio, Wi-Fi, radar, satellite communication and optical networks rely on electromagnetic waves.
- Different frequencies, same wave principles.

6 Waves in Many Media

Fluids Strings Membranes Solids



Waves occur in fluids, strings, membranes and solids – in countless forms.

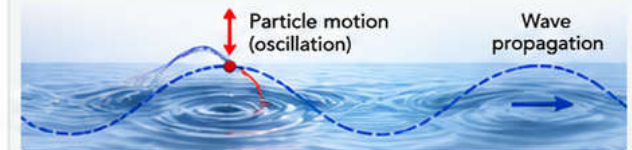
Common Questions in Wave Physics

- How fast does the wave travel?
- What is its wavelength?
- What is its frequency?
- How much energy does it carry?
- What happens when it meets a boundary?
- Can reflection produce resonance?
- Can two waves interfere?



Key Idea

Waves transport energy and information, but the medium does not travel with the wave over long distances.



When a water wave passes, each water particle mainly oscillates around its local position.



Why it matters for engineers:

Understanding wave behaviour allows us to design better systems, avoid harmful resonance, communicate across distances, control vibrations and acoustics, and use waves to sense and explore the world.



Once we understand the general mathematical structure, we can apply it across many fields.

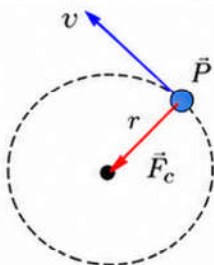


REVIEW: Circular and Rotational Motion



Last lecture we studied how objects move in circles and how torques cause rotation. Here is a quick review of the key concepts.

1. Circular Motion Basics

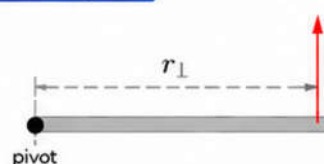


- To move in a circle, an object needs a centripetal force.
- This force points toward the center.
- Its magnitude is

$$F_c = m \frac{v^2}{r} = mv\omega^2 r$$

$$\omega = \frac{v}{r} \quad (\text{angular speed})$$

2. Torque



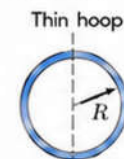
- Torque is the rotational effect of a force about an axis.
- $r_{\perp}$: perpendicular distance from axis to line of action of $\vec{F}$.
- Units: N·m

$$\vec{\tau} = \vec{r} \times \vec{F}$$

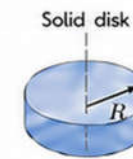
$$\tau = r_{\perp} F \sin \phi$$

3. Moment of Inertia

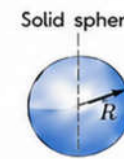
Depends on how mass is distributed relative to the axis.



$$I = MR^2$$



$$I = \frac{1}{2} MR^2$$



$$I = \frac{2}{5} MR^2$$

$$\text{Parallel axis theorem: } I = I_{cm} + Md^2$$

4. Rotational Dynamics

Torque causes angular acceleration.

$$\sum \tau = I\alpha$$

Angular kinematics (constant α):

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Definitions

θ : angular position (rad)

ω : angular velocity (rad/s)

α : angular acceleration (rad/s²)

Units:

θ (rad), ω (rad/s), α (rad/s²)

5. Angular Momentum

Angular momentum describes rotational motion.

$$\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}$$

For a system:

$$\vec{L}_{net} = I_{tot} \vec{\omega}$$

Units: kg·m²/s

Conservation

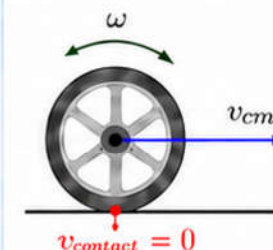
If no external torque acts on a system,

$$\vec{\tau}_{ext} = 0$$



$$\vec{L} = \text{constant}$$

6. Rolling Without Slipping



Translation and rotation are linked.

$$v_{cm} = \omega R$$

$$a_{cm} = \alpha R$$

The contact point is instantaneously at rest relative to the surface.



These concepts are essential for understanding the motion of real-world systems—and they will also help us understand waves and oscillations.



Key takeaway: For circular and rotational motion, forces and torques cause changes in motion. Next, we will see how periodic motion leads to waves that carry energy and information.



Coming up:

We begin with the basic idea of what a wave is.



Review: Circular and Rotational Motion

Angular quantities

Angular position

$$\theta$$

Angular velocity

$$\omega = \frac{d\theta}{dt}$$

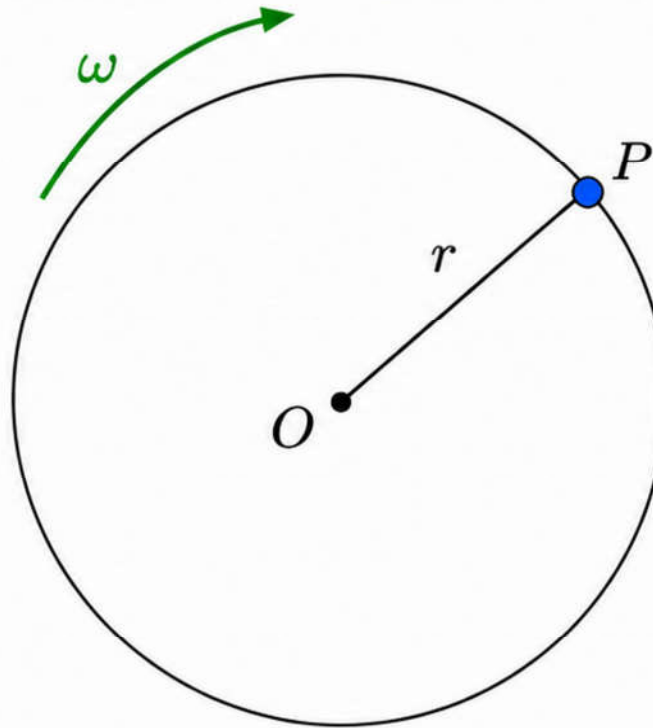
Angular acceleration

$$\alpha = \frac{d\omega}{dt}$$

For constant α :

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$



Linear–angular relations at radius r

Arc length

$$s = r\theta$$

Tangential speed

$$v_t = r\omega$$

Tangential acceleration

$$a_t = r\alpha$$

Radial acceleration

$$a_r = \frac{v^2}{r} = \omega^2 r$$

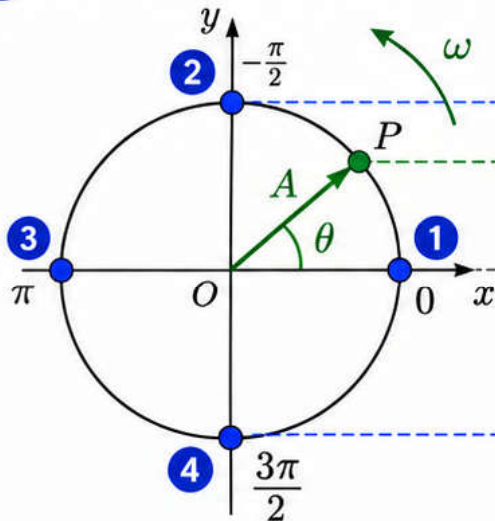


Key idea: Rotation is described by angular quantities; motion at radius r connects them to linear speed and acceleration

From Rotation to Oscillation

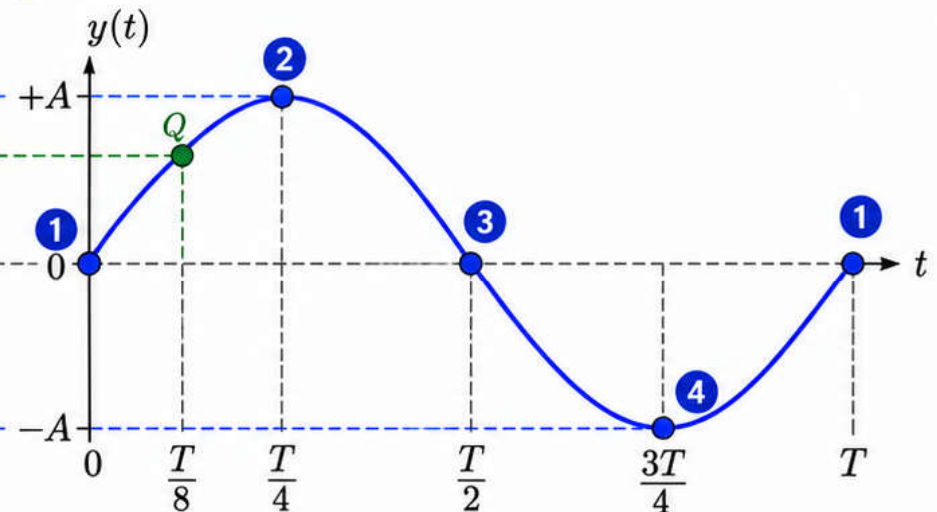
The vertical projection of uniform circular motion traces a sinusoidal displacement.

1 Uniform circular motion



$$\theta = \omega t$$

2 Displacement versus time



$$y(t) = A \sin(\omega t)$$

$$T = \frac{2\pi}{\omega} \quad \Bigg| \quad f = \frac{1}{T}$$



Matching labels indicate the same phase; horizontal guides indicate the same displacement.

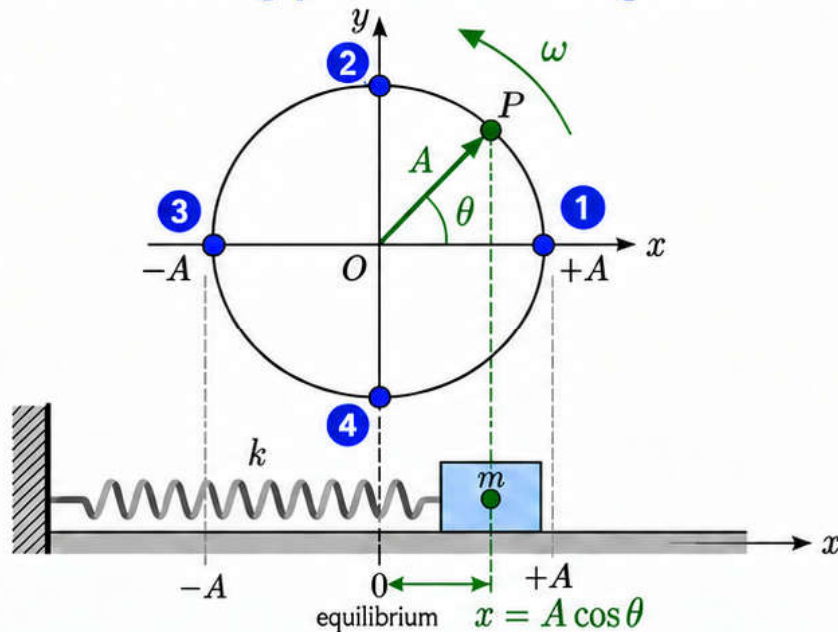


Main idea: Uniform circular motion and its projection share amplitude A , angular frequency ω , and period T .

Circular Motion and a Spring–Mass Oscillator

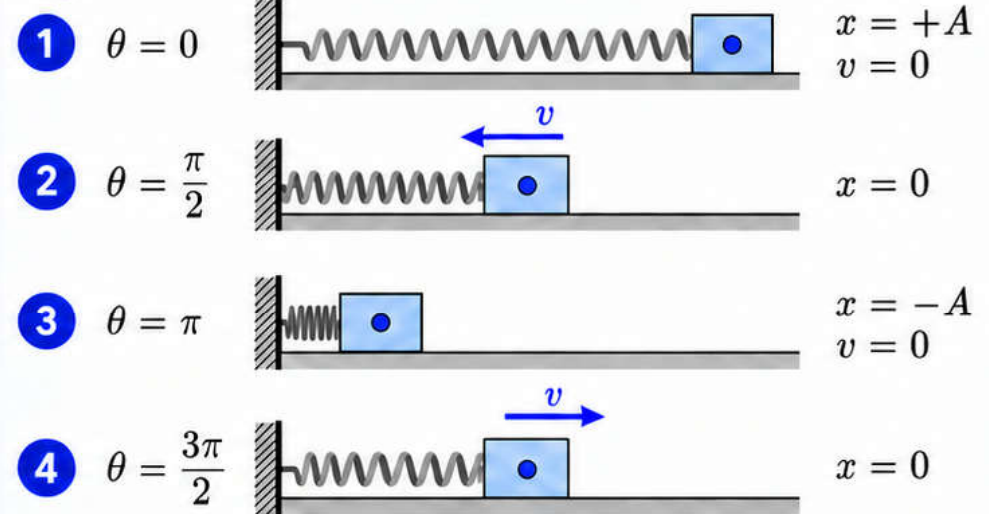
The horizontal projection of uniform circular motion has the same displacement as the oscillating mass.

1 From a rotating point to a moving mass



$$\theta = \omega t \quad x(t) = A \cos(\omega t)$$

2 Four corresponding states



One full cycle: 1 → 2 → 3 → 4 → 1

Schematic snapshots

$$\omega = \sqrt{\frac{k}{m}} \quad \text{Ideal spring; no damping}$$

i The circle is a mathematical model; the mass moves only along the horizontal axis.

Main idea: The rotating point and the mass share the same horizontal displacement x .



What Is a Wave?

A disturbance propagates through a medium—or through space—carrying energy and information.



Key idea: In a wave, it is the **disturbance** that travels, **not the matter**.

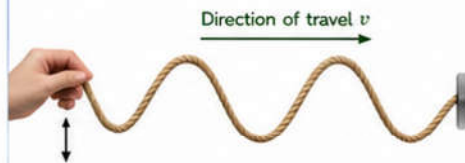
- A wave is a traveling disturbance that transfers energy from one place to another.
- The particles of the medium (or the fields in space) oscillate about their equilibrium positions but do not travel with the wave.
- Waves can be mechanical (need a medium) or electromagnetic (do not need a medium).

Remember:

Energy and information move with the wave.

Examples of Waves in Nature and Technology

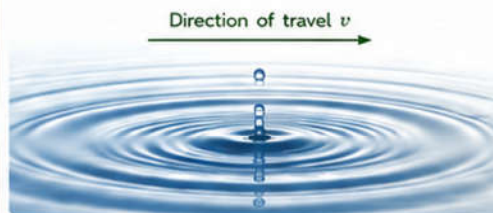
1. Wave on a String (Rope)



The rope moves up and down, but the wave travels to the right.

Disturbance travels → Energy transfers

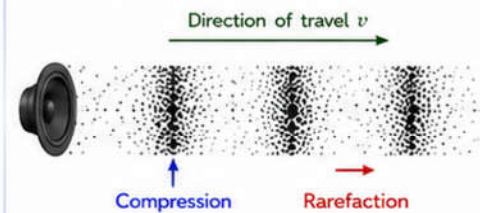
2. Water Waves



Water moves up and down, but the wave spreads outward.

Disturbance travels → Energy transfers

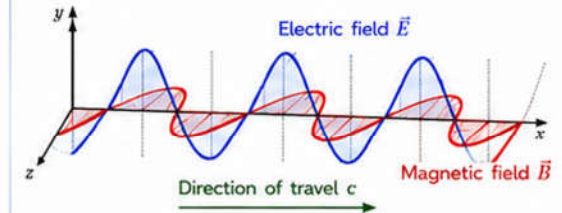
3. Sound Wave (Longitudinal)



Air particles oscillate back and forth parallel to the direction of travel.

Disturbance travels → Energy transfers

4. Light (Electromagnetic Wave)



Electric and magnetic fields oscillate perpendicular to each other and to the direction of travel.

Disturbance travels → Energy transfers



Takeaway

A wave transports energy (and possibly information) through a medium or space. The local elements of the medium **oscillate**, but the wave itself **moves on**.



Think about it

When you are at the beach and see an ocean wave pass, do you move forward with the wave? Why or why not?



Today's goal: Understand what a wave is and how it differs from the motion of the medium.



Next: Oscillations



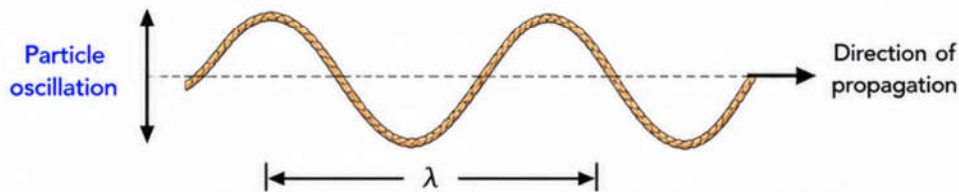
Types of Mechanical Waves



Mechanical waves are disturbances that propagate through a material medium. The medium's particles oscillate about equilibrium while energy travels with the disturbance.

1. TRANSVERSE WAVES

Particle displacement is **perpendicular** ($\perp$) to the direction of propagation.

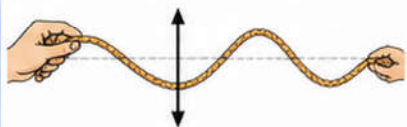


Key features

- Oscillation direction $\perp$ propagation direction.
- Crests and troughs occur in sinusoidal examples.
- Examples: waves on a stretched string; seismic S-waves.

Examples

Wave on a stretched string

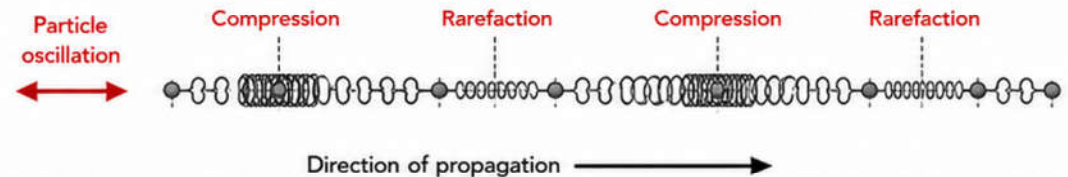


Seismic S-wave



2. LONGITUDINAL WAVES

Particle displacement is **parallel** ($\parallel$) to the direction of propagation.

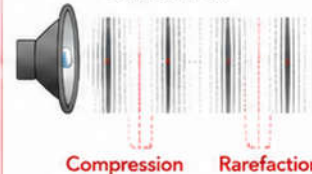


Key features

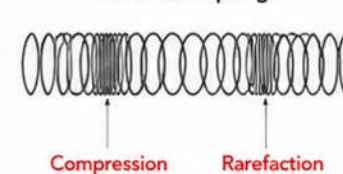
- Oscillation direction $\parallel$ propagation direction.
- Alternating compressions and rarefactions.
- Examples: sound in air; waves in a spring; seismic P-waves.

Examples

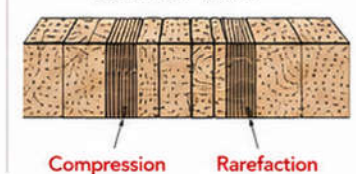
Sound in air



Wave in a spring



Seismic P-wave



Important: Surface water waves are not purely transverse—the particles usually follow circular or elliptical paths.



Takeaway: Classify a mechanical wave by comparing particle displacement with the direction of propagation: **transverse** $\perp$, **longitudinal** $\parallel$.



Mechanical vs. Electromagnetic Waves



Waves come in two broad classes: **mechanical waves** and **electromagnetic waves**. They differ in what oscillates, whether a medium is required, and how they can travel.

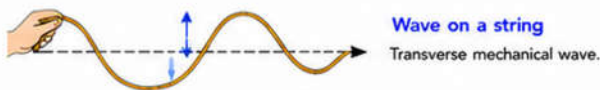
1. MECHANICAL WAVES

Waves that **require a material medium**. They arise from the oscillation of particles of matter.

What oscillates?

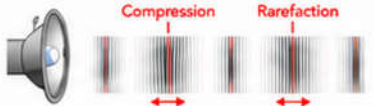
Particles of matter (atoms, molecules, elements).

Examples



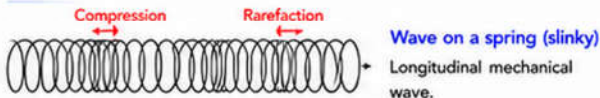
Wave on a string

Transverse mechanical wave.



Sound in air

Longitudinal mechanical wave.



Wave on a spring (slinky)

Longitudinal mechanical wave.



Key point

- Mechanical waves require a material medium and involve oscillating particles of matter.
- Mechanical waves can be transverse or longitudinal.
- Mechanical-wave speed depends on properties of the medium.
- Mechanical waves cannot travel in vacuum.

2. KEY DIFFERENCES

Property	Mechanical Waves	Electromagnetic Waves
Medium required?	Yes Require a material medium (solid, liquid, or gas).	No Do not require a medium. Can travel in vacuum.
What oscillates?	Particles of matter (atoms, molecules, elements).	Electric and magnetic fields (E and B).
Direction of oscillation	Parallel (longitudinal) or perpendicular (transverse) to propagation.	For an ideal plane wave: $E \perp B \perp$ direction of propagation.
Speed	Depends on properties of the medium.	In vacuum: $c = 3.00 \times 10^8$ m/s (same for all EM waves).
Can travel in vacuum?	No	Yes
Examples	Sound, waves on a string, water waves, seismic waves.	Radio waves, visible light, X-rays, microwaves, gamma rays.

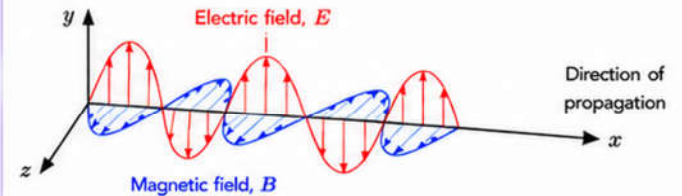
3. ELECTROMAGNETIC WAVES

Waves that **do not require a material medium**. They consist of oscillating **electric and magnetic fields**.

What oscillates?

Electric field (E) and magnetic field (B).

How it looks



Key point

- In vacuum, all electromagnetic waves travel at $c = 3.00 \times 10^8$ m/s.
- Electromagnetic waves can travel in vacuum.

Examples



Radio waves



Microwaves



Visible light



X-rays



Gamma rays



4. SUMMARY

- Mechanical waves require a material medium and involve oscillating particles of matter.
- Mechanical waves can be transverse or longitudinal.
- Mechanical-wave speed depends on properties of the medium.
- Mechanical waves cannot travel in vacuum.

Remember:

If it needs a medium $\rightarrow$ mechanical wave.
If it can travel in vacuum $\rightarrow$ electromagnetic wave.

Oscillation vs. Propagation

Transverse:	$\perp$
Longitudinal:	$\parallel$

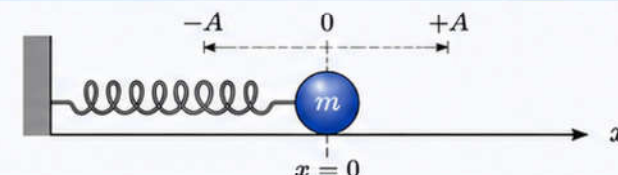


Oscillations: The Building Block of Waves

A wave is built from **oscillations**. To understand waves, we first understand oscillatory motion.

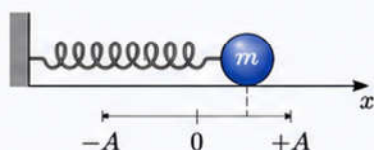
Definition: An oscillation is a repetitive back-and-forth motion of an object about an **equilibrium position**.

- A restoring force acts to bring the object back toward equilibrium when it is displaced.
- If friction is negligible, the motion is periodic and repeats itself.



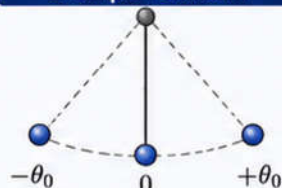
Examples of Oscillatory Motion

1. Mass-Spring System



The mass oscillates about equilibrium as the spring exerts a restoring force $F = -kx$.

2. Simple Pendulum



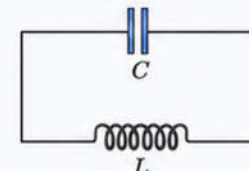
For small angles, the pendulum undergoes approximately simple harmonic motion.

3. Tuning Fork



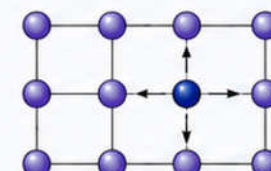
The prongs vibrate back and forth at a nearly fixed natural frequency when struck.

4. Electric Circuit (LC)



Charge and current oscillate in the circuit. Energy alternates between the capacitor's electric field and the inductor's magnetic field.

5. Atoms in a Solid



Atoms vibrate about their equilibrium positions in a crystal lattice.

Key Characteristics of Oscillatory Motion

Amplitude (A):

maximum displacement from equilibrium.

Period (T):

time for one complete oscillation (s).

Frequency (f):

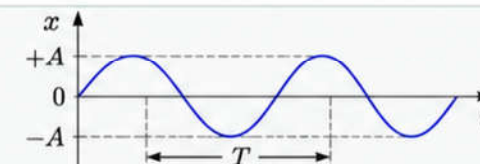
number of oscillations per second (Hz); $f = 1/T$.

Angular frequency (ω):

$\omega = 2\pi f = \frac{2\pi}{T}$ (rad/s).

Phase (ϕ):

specifies the state of motion at a given time.



Takeaway:

Oscillations involve repetitive motion about equilibrium. They provide the “**heartbeat**” from which waves are created.



Think about it:

What is the restoring force in each example above?
How would the motion change if damping (friction) were present?



Today's goal: Understand oscillatory motion and the key quantities used to describe it.



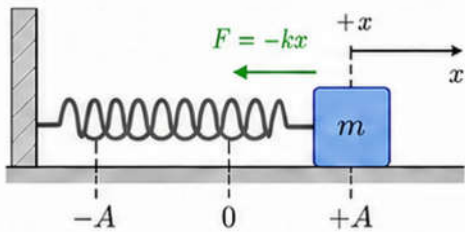
Next: Wavelength and Period

Simple Harmonic Motion and Coupled Oscillators



For simple harmonic motion, the restoring force is proportional to displacement and directed toward equilibrium.

1. The standard model: Spring–mass oscillator



$$F = -kx$$

$$m \frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Solution (simple harmonic motion):

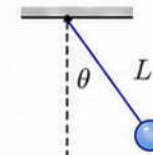
$$x(t) = A \cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{k}{m}}$$

A = amplitude (maximum displacement)
 ω = angular frequency (rad/s)
 ϕ = phase constant (set by initial conditions)

2. Other examples of oscillators

Simple pendulum



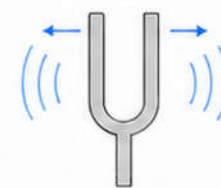
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

(small-angle approximation)

$$\theta(t) = \Theta \cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{g}{L}}$$

Tuning fork



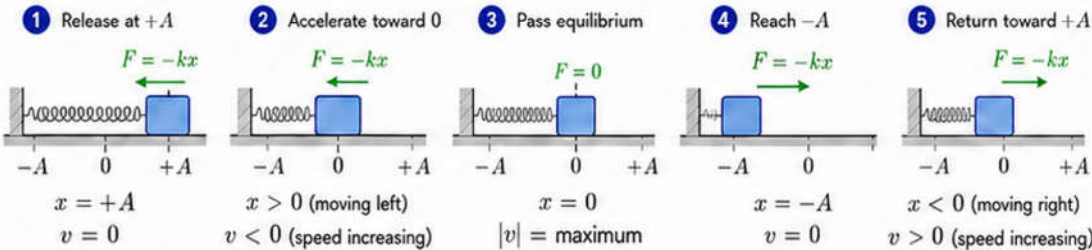
Prong displacement:
 $x(t) \approx A \cos(\omega t + \phi)$
 ω is set mainly by the
 fork's geometry and
 material.

Physical pendulum



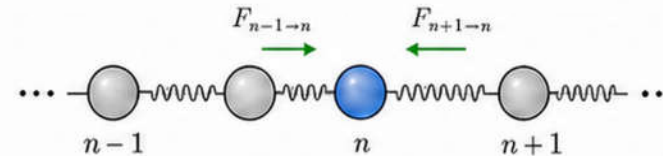
$\theta(t) \approx \Theta \cos(\omega t + \phi)$
 ω depends on mass
 distribution and pivot
 position.

3. One complete oscillation



In an ideal oscillator, kinetic energy and elastic potential energy continuously exchange while total mechanical energy remains constant.

4. From oscillators to waves



Oscillation + coupling → wave propagation

Key idea:
 When one oscillator moves, it exerts forces on its neighbors. The disturbance then propagates through the system.



Take-home message: Simple harmonic oscillators are fundamental models for wave motion. Coupling lets energy and information propagate from one location to the next.



Why it matters: Mechanical waves arise from coupled particles; electromagnetic waves arise from coupled electric and magnetic fields.



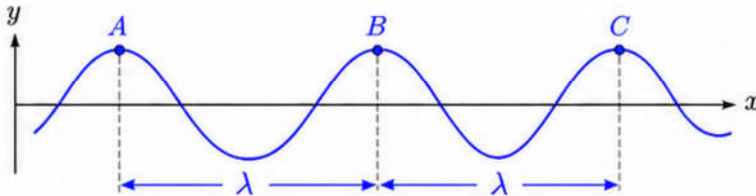
Wavelength and Period: Two Fundamental Quantities



Two key quantities describe any periodic wave: **wavelength λ** (a spatial measure) and **period T** (a time measure). They tell us how far the wave pattern repeats in space and how long it takes to repeat in time.

Wavelength λ (spatial)

The wavelength is the shortest distance between two points in the **same phase**.



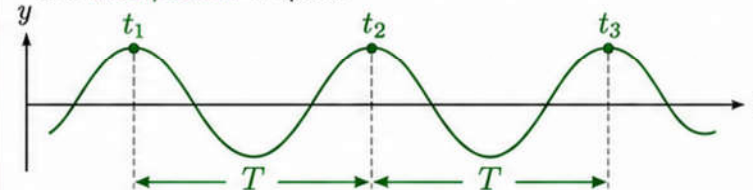
- Distance between successive crests (or troughs).
- Units: meters (m).
- Represents the spatial repeat length of the wave.

Also between:

- crest to crest
- trough to trough
- any two nearest points in the same phase.

Period T (temporal)

The period is the time required for one complete cycle of oscillation at a **fixed position** in space.



- Time for one complete cycle.
- Units: seconds (s).
- Represents the temporal repeat time of the wave.

Remember:

At a fixed position, the same motion repeats every T .

λ tells us about **space** (how far apart equal-phase features are).

T tells us about **time** (how long one cycle takes).

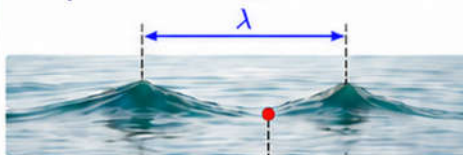


They are connected, but different.

When a periodic wave travels at speed v :

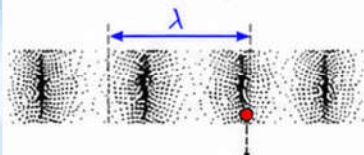
$$v = \frac{\lambda}{T} = f\lambda.$$

Examples Water wave (surface elevation)



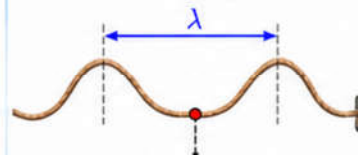
Measure T as the time interval between successive peaks at the fixed **red** point.

Sound wave (longitudinal)



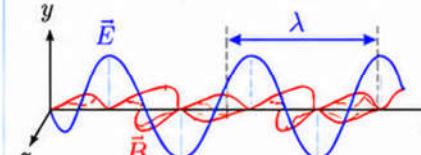
Measure T as the time interval between successive compressions at the fixed **red** point.

Sine wave on a string (transverse)



Measure T as the time interval between successive peaks at the fixed **red** point.

Electromagnetic wave (transverse)



Measure T as the time interval between successive peaks of $\vec{E}$ at the fixed **red** point.

Important distinction

λ is the spatial repeat length;
 T is the temporal repeat time.
For a traveling periodic wave, the wave pattern advances one wavelength during one period:

$$\lambda = vT.$$

The medium's particles generally oscillate locally rather than moving with the wave.



Today's goal: Understand wavelength λ and period T , identify them in different waves, and connect them through wave speed.



Next: Frequency and the relationship $f = 1/T$



Frequency: How Often?

Frequency tells us how many cycles occur each second.



Key idea: The frequency f is the number of complete **oscillations** (cycles) per second at a fixed point in space. It tells us how rapidly the oscillation repeats.

DEFINITION

Frequency f is defined as

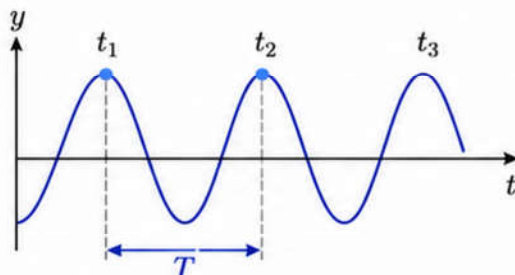
$$f = \frac{1}{T}$$

where

- T = period (s)
- f = frequency (Hz = s⁻¹)
- 1 Hz means 1 cycle per second

Example:

If the period of a wave is $T = 0.50$ s, then the frequency is $f = 1/T = 2.0$ Hz.



UNITS

- SI unit: hertz (Hz)
- 1 Hz = 1 cycle per second

Unit	Symbol	In hertz
Hertz	Hz	1 Hz
Kilohertz	kHz	10^3 Hz = 1,000 Hz
Megahertz	MHz	10^6 Hz = 1,000,000 Hz
Gigahertz	GHz	10^9 Hz = 1,000,000,000 Hz

Real-world sense



For a pure sound tone, higher frequency means higher pitch.

FROM PERIOD TO FREQUENCY

If a wave has period T , then in a time t it completes N cycles:

$$N = \frac{t}{T}$$



Therefore,

$$f = \frac{N}{t} = \frac{1}{T}$$

EXAMPLE

A sound wave has $f = 440$ Hz. What is its period?

$$T = \frac{1}{f} = \frac{1}{440} \text{ s} \approx 2.27 \times 10^{-3} \text{ s} = 2.27 \text{ ms}$$



Takeaway

- Frequency f tells how many cycles occur each second.
- It is the repetition rate of the oscillation.



Think about it

Which has the higher frequency?

- a 2 Hz wave or a 50 Hz wave?
- a wave with $T = 10$ ms or $T = 1$ ms?



Check your understanding

A wave has frequency 25 Hz. How many cycles occur in 3.0 seconds?



Common mistake

Confusing frequency f (cycles per second) with period T (seconds per cycle).



Today's goal: Understand frequency: its meaning, units, and relationship to period.



Next: Angular Frequency ω



Wave Number k : Spatial Frequency of a Wave

Wave number k describes how rapidly the phase of a wave varies in space.



Key idea: Wave number k is the rate at which the phase changes with position. It determines how many radians (or cycles) fit into a given distance.

Definition

The wave number k is defined as

$$k = \frac{2\pi}{\lambda}$$

where

- k = wave number (rad/m)
- λ = wavelength (m)
- 2π rad = one full cycle in phase

Units:

k has units of radians per meter (rad/m).
Radians are dimensionless.

What Does k Mean?

k tells us how quickly the phase advances as we move in the $+x$ direction.

$$\text{phase at position } x: \phi(x) = kx + \phi_0$$

After moving a distance Δx , the phase changes by

$$\Delta\phi = k \Delta x$$

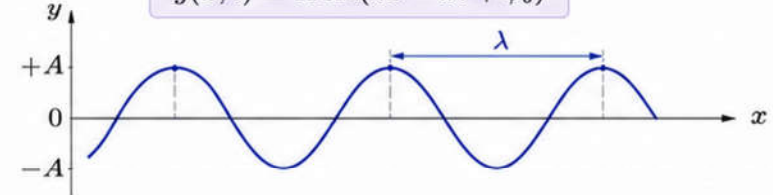
In particular:

- If $\Delta x = \lambda$, then $\Delta\phi = 2\pi$ (one full cycle).
- If $\Delta x = \frac{\lambda}{2}$, then $\Delta\phi = \pi$ (half a cycle).

Relation to Spatial Patterns

A sinusoidal wave traveling in $+x$ can be written as

$$y(x, t) = A \sin(kx - \omega t + \phi_0)$$



Larger $k \Rightarrow$ shorter wavelength $\Rightarrow$ more cycles per meter.
Smaller $k \Rightarrow$ longer wavelength $\Rightarrow$ fewer cycles per meter.

Connection to Other Quantities

In terms of f and v

Since $v = f\lambda$ and $k = 2\pi/\lambda$,

$$k = \frac{2\pi f}{v} = \frac{\omega}{v}$$

(using $\omega = 2\pi f$)

In terms of ω and λ

Using $\omega = 2\pi f$,

$$k = \frac{\omega}{v}$$

Phase velocity

The wave speed (phase velocity) is

$$v = \frac{\omega}{k}$$

Examples

1. Water wave: $\lambda = 0.50$ m

$$k = \frac{2\pi}{0.50 \text{ m}} = 12.6 \text{ rad/m}$$

2. Sound in air: $f = 440$ Hz, $v = 343$ m/s

$$k = \frac{2\pi f}{v} = \frac{2\pi(440)}{343} = 8.06 \text{ rad/m}$$

Wavenumber vs. Frequency

Quantity	Describes...
f (Hz)	how often in time (temporal cycles)
k (rad/m)	how often in space (spatial phase rate)
ω (rad/s)	how fast in time (temporal phase rate)
v (m/s)	how fast it travels (propagation speed)



Takeaway

- Wave number $k = 2\pi/\lambda$ measures spatial phase rate in radians per meter.
- It links space (position) to phase.



Think about it

If the wavelength of a wave is halved, what happens to the wave number k ?
What happens to the phase change over a fixed distance?



Check your understanding

A wave has $k = 7.50$ rad/m.
What is its wavelength λ ?
How many radians of phase change occur over $\Delta x = 0.20$ m?



Common misconception

Thinking that k is the number of wavelengths per meter. The number of wavelengths per meter is $1/\lambda = k/(2\pi)$, not k .



Today's goal: Understand wave number k , its meaning, and how it relates to wavelength, frequency, angular frequency, and wave speed.



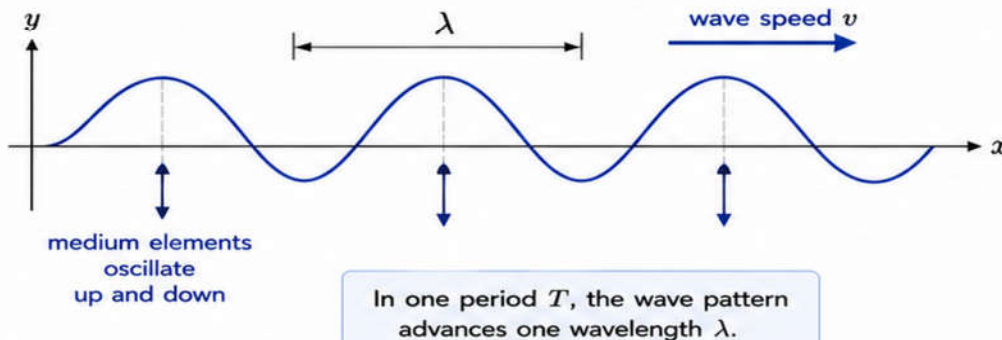
Next: Wave Speed



Wave Speed: Connecting Space and Time

One propagation speed, expressed with wavelength and frequency — or with their angular counterparts.

1. Wave Pattern in Space



2. The Relations

Spatial description (space)

λ = wavelength (m)

k = wave number (rad/m)

$$k = \frac{2\pi}{\lambda}$$

Temporal description (time)

T = period (s)

f = frequency (Hz)

ω = angular frequency (rad/s)

$$f = \frac{1}{T}$$

$$\omega = 2\pi f$$

Connection (one propagation speed)

$$v = \frac{\lambda}{T} = f\lambda = \frac{\omega}{k}$$

$$\frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = f\lambda$$

Units: v (m/s) | λ (m) | f (Hz) | ω (rad/s) | k (rad/m)

3. Physical Meaning

- v is the propagation speed of a point of constant phase, and of wave energy/information.
- The particles of a material medium generally oscillate about equilibrium; they do not travel with the wave.
- The medium and its conditions determine the wave speed.
- The source fixes f ; if a wave enters another medium, f remains unchanged and λ changes so that $v = f\lambda$.

4. Example

A water wave has $\lambda = 0.60$ m and $f = 2.0$ Hz. Find v , ω , and k .

$$v = f\lambda = (2.0 \text{ Hz})(0.60 \text{ m}) = 1.2 \text{ m/s}$$

$$\omega = 2\pi f = 4\pi \text{ rad/s}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.60 \text{ m}} = \frac{10\pi}{3} \text{ rad/m}$$

$$\therefore \frac{\omega}{k} = \frac{4\pi}{10\pi/3} = 1.2 \text{ m/s}$$

5. Medium Examples (speed v)



Stretched string

$$v = \sqrt{\frac{T_{\text{tension}}}{\mu}}$$

T_{tension} = tension (N)
 μ = linear mass density (kg/m)



Sound in air (at 20 °C)

$$v \approx 343 \text{ m/s}$$



Electromagnetic wave in vacuum

$$c = 3.00 \times 10^8 \text{ m/s}$$



Deep-water gravity wave (phase speed)

$$v_p = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

$g = 9.81 \text{ m/s}^2$



Qualification: The frequency-independent speed applies only in a nondispersive medium or approximately over a narrow frequency range.



Key Takeaway

Wave speed links the spatial cycle of a wave to its temporal cycle.



Today's goal: Understand how wave speed connects wavelength, frequency, and angular quantities.



Next: Phase Angle ϕ



Phase Angle ϕ : Describes the Stage of a Wave

The phase angle tells us where we are in the cycle of a periodic motion.

1. Traveling Wave

A traveling wave can be written as

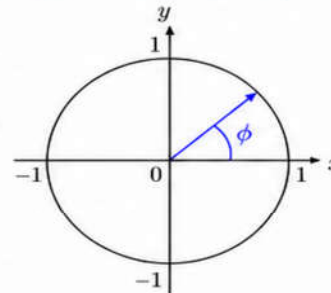
$$y(x, t) = A \sin(kx - \omega t + \phi)$$

where ϕ is the initial phase angle (in radians) that locates the state within the cycle at a chosen position x and time t .

A	= amplitude	(meters, m)
k	= wave number	(radians per meter, rad/m)
ω	= angular frequency	(radians per second, rad/s)
x	= position	(meters, m)
t	= time	(seconds, s)
ϕ	= phase angle (initial phase)	(radians, rad)

2. Phase Angle on the Unit Circle

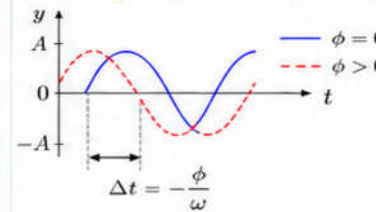
- ϕ is measured counterclockwise from the positive x -axis.



- The normalized displacement is $\frac{y}{A} = \sin \phi$.

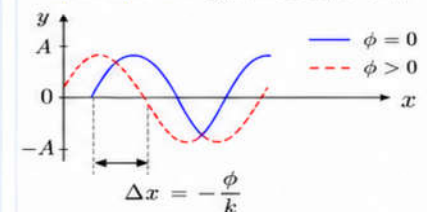
3. Phase Shift

Time signal: $y(t) = A \sin(\omega t + \phi)$



For $\phi > 0$, the waveform is shifted left (earlier) by $\Delta t = -\frac{\phi}{\omega}$.

Spatial profile: $y(x) = A \sin(kx + \phi)$



For $\phi > 0$, the waveform is shifted left by $\Delta x = -\frac{\phi}{k}$.

4. Phase Comparison

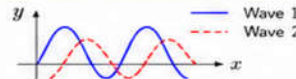
In phase

$\Delta\phi = 2\pi m$, $m \in \mathbb{Z}$ (equivalently, $\Delta\phi \equiv 0 \pmod{2\pi}$): crests and troughs coincide.



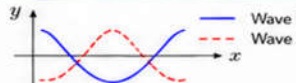
Out of phase

$\Delta\phi \neq 0 \pmod{2\pi}$: crests and troughs do not coincide.



Completely opposite phase

$\Delta\phi = (2m + 1)\pi$, $m \in \mathbb{Z}$: crest of one aligns with trough of the other.



5. Special Phase Angles

ϕ (rad)	ϕ (deg)	$\sin \phi$	Meaning
0	0°	0	upward zero crossing
$\frac{\pi}{2}$	90°	1	positive maximum
π	180°	0	downward zero crossing
$\frac{3\pi}{2}$	270°	-1	negative maximum
2π	360°	0	one full cycle / same state as 0

6. Interpretation

- Phase is defined modulo 2π :

ϕ and $\phi + 2\pi m$ ($m \in \mathbb{Z}$) represent the same physical state.

- For the forms $y(t) = A \sin(\omega t + \phi)$ and $y(x) = A \sin(kx + \phi)$:

If $\phi > 0$, the waveform is shifted left (earlier) by $\Delta t = -\frac{\phi}{\omega}$ in time and $\Delta x = -\frac{\phi}{k}$ in space, as shown above.



Check Your Understanding

For $y(x, t) = A \sin(kx - \omega t + \phi)$, with $\phi = \pi/3$ and $kx - \omega t = 0$, what is y ?

$$y = \frac{\sqrt{3}}{2} A.$$



Think About It

Two waves have the same amplitude and frequency but a phase difference $\Delta\phi = \pi/2$. How are their motions related?



Key Takeaway

The phase angle locates a wave within its cycle and controls relative shifts between otherwise identical waves.



Common Mistake

Phase ϕ is an angle in radians—not a position x or a time t .



Today's goal: Understand the meaning of phase angle ϕ , how it affects a wave, and how to interpret phase differences.



Next: Quick Reference Guide



Quick Reference Guide: Wave Quantities and Relations

The key quantities that describe a sinusoidal wave — and how they are related.



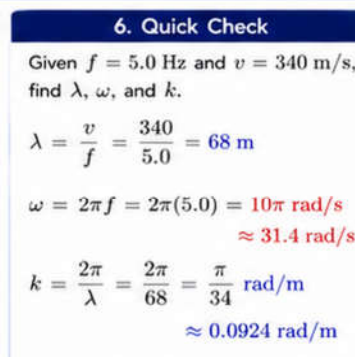
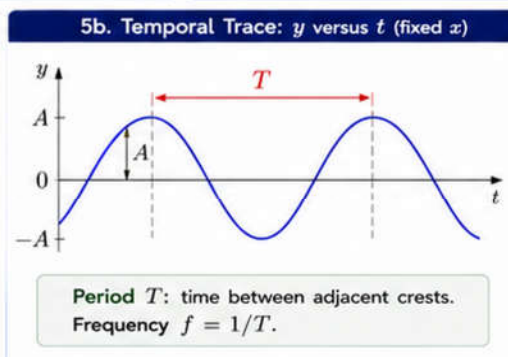
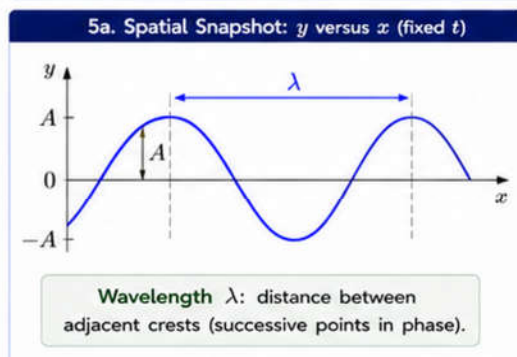
Key idea: A sinusoidal traveling wave is described by amplitude and phase together with linked spatial and temporal scales. Appropriate independent data — plus the propagation relation or medium properties — are needed to determine the remaining quantities.

1. Key Wave Quantities			
Quantity	Symbol	Definition	SI Unit
Wavelength	λ	Distance between successive points in phase	m
Period	T	Time for one cycle	s
Frequency	f	Cycles per second	$\text{Hz} = \text{s}^{-1}$
Angular frequency	ω	Temporal phase rate; $\omega = 2\pi f = 2\pi/T$	rad/s
Wave number	k	Spatial phase rate; $k = 2\pi/\lambda$	rad/m
Phase angle	ϕ	Stage/offset within cycle	rad
Amplitude	A	Maximum displacement from equilibrium	Same as displacement

2. Fundamental Relations		
Frequency	$f = \frac{1}{T}$	f : cycles per second
Angular frequency	$\omega = 2\pi f = \frac{2\pi}{T}$	ω : radians per second
Wave number	$k = \frac{2\pi}{\lambda}$	k : radians per meter
Wave speed (propagation)	$v = f\lambda = \frac{\omega}{k}$	v : wave speed (m/s)
Useful derived forms		
$\lambda = \frac{v}{f}$	$T = \frac{\lambda}{v}$	$\omega = kv$
$f = \frac{\omega}{2\pi} = \frac{vk}{2\pi}$		

3. Traveling-Wave Sign Convention	
For $k > 0$, the sign before ωt determines propagation direction.	
$+x$ direction:	$y(x, t) = A \sin(kx - \omega t + \phi)$
$-x$ direction:	$y(x, t) = A \sin(kx + \omega t + \phi)$

4. Phase Visualization	
Unit circle	Phase shift in $y(x, t) = A \sin(kx + \phi)$
	Positive ϕ shifts the profile left by $\Delta x = -\phi/k$.
$y/A = \sin \phi$	



7. Common Mistake

ω and k are angular phase rates, not “per-cycle” quantities; keep the factors of 2π .



Key Takeaway: Wave quantities are linked through exact relations, but you must have a sufficient independent set of measurements or known properties — together with the appropriate propagation relation or medium properties — to determine the rest.



Today's Goal: Understand the key wave quantities, their meanings and units, and the exact relations that connect them.



Next: Worked Example 1



Worked Example 1: Basic Wave Parameters

From one spatial scale and one temporal scale to the related wave parameters.



Key idea: Given wavelength λ and frequency f , we have one spatial scale and one temporal scale. These are sufficient to determine T , ω , k and v – but not the amplitude A or initial phase ϕ .

1. Problem Statement

A sinusoidal traveling wave on a string has

- Wavelength: $\lambda = 0.80$ m
- Frequency: $f = 25$ Hz

Determine:

- Period T
- Angular frequency ω
- Wave number k
- Wave speed v

Assume propagation in the $+x$ direction.

2. Step-by-Step Solution

1 Period:

$$T = \frac{1}{f} = \frac{1}{25 \text{ s}^{-1}} = 0.040 \text{ s}$$

2 Angular frequency:

$$\omega = 2\pi f = 2\pi(25 \text{ s}^{-1}) = 50\pi \text{ rad/s} \approx 157.08 \text{ rad/s}$$

3 Wave number:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.80 \text{ m}} = 2.5\pi \text{ rad/m} \approx 7.85 \text{ rad/m}$$

4 Wave speed:

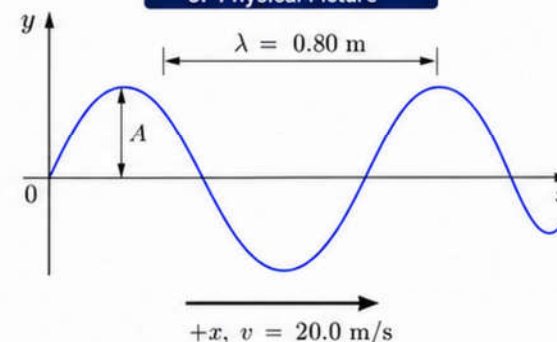
$$v = f\lambda = (25 \text{ s}^{-1})(0.80 \text{ m}) = 20.0 \text{ m/s}$$



Consistency check:

$$v = \frac{\omega}{k} = \frac{50\pi}{2.5\pi} = 20.0 \text{ m/s} \quad \checkmark$$

3. Physical Picture



4. Summary Table of Results

Quantity	Symbol	Formula	Value	SI unit
Wavelength	λ	given	0.80	m
Frequency	f	given	25	Hz
Period	T	$1/f$	0.040	s
Angular frequency	ω	$2\pi f$	$50\pi \approx 157.08$	rad/s
Wave number	k	$2\pi/\lambda$	$2.5\pi \approx 7.85$	rad/m
Wave speed	v	$f\lambda$ or ω/k	20.0	m/s

Interpretation of the Results

- The wave completes 25 cycles per second.
- One cycle lasts 0.040 s.
- The phase changes by 2π radians over 0.80 m in space and by 2π radians every 0.040 s in time.
- The wave pattern propagates at 20.0 m/s.
- The amplitude A and initial phase ϕ remain unspecified by the given λ and f .

5. General Takeaway

To determine this kinematic set, use one independent spatial scale and one independent temporal scale, or wave speed together with one of them.

Redundant pairs such as (λ, k) or (f, ω) do not provide two independent pieces of information.



Think about it

If f doubles while λ remains fixed, what happens to v ?



Common mistake

Hz and s^{-1} are equivalent units, so no numerical conversion factor is needed. Keep the factors of 2π in $\omega = 2\pi f$ and $k = 2\pi/\lambda$.



Check your understanding

Given $\lambda = 0.60$ m and $v = 18$ m/s, find f , T , ω , and k .



Take care

The amplitude A and initial phase ϕ cannot be determined from λ , f , and v alone. Additional information is required.



Today's goal: Practice computing all key wave parameters from one spatial scale and one temporal scale.



Next: Traveling Waves



Traveling Waves: Waves That Move Through Space

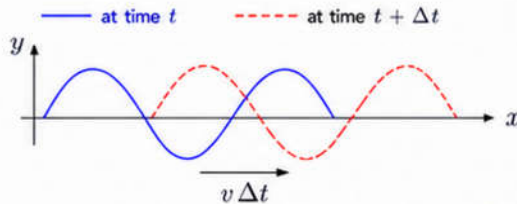
A traveling wave carries a disturbance and energy through space while medium elements usually oscillate locally.



Key idea: In an ideal linear traveling-wave model, the wave pattern propagates while each material element oscillates about its equilibrium position.

1. Wave moving in $+x$ (to the right)

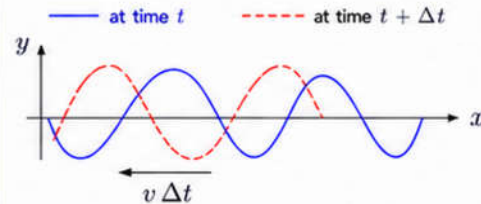
$$y(x, t) = A \sin(kx - \omega t + \phi)$$



The later pattern is shifted right by $v \Delta t$.

2. Wave moving in $-x$ (to the left)

$$y(x, t) = A \sin(kx + \omega t + \phi)$$



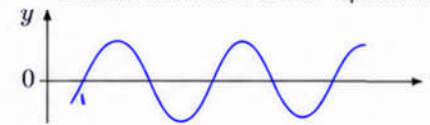
The later pattern is shifted left by $v \Delta t$.

3. Particle motion versus wave motion

At fixed position $x = x_0$ (particle motion in time)

$$y(x_0, t) = A \sin(kx_0 - \omega t + \phi)$$

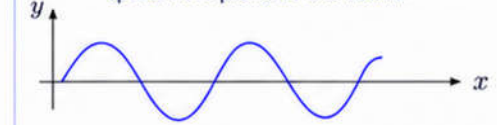
Sinusoidal motion in time about equilibrium.



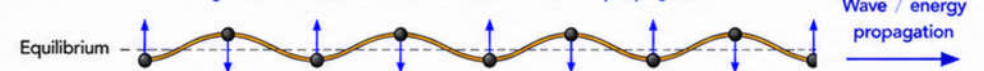
At fixed time $t = t_0$ (wave profile in space)

$$y(x, t_0)$$

Spatial wave profile at one instant.



String model (material elements oscillate; wave propagates)



Material points oscillate up and down about fixed positions; the wave pattern (and energy/information) moves right.

6. Direction rule (from constant phase)

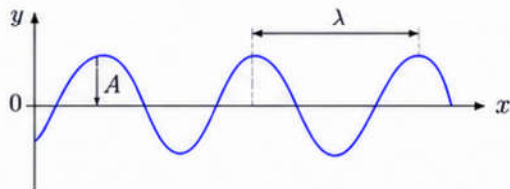
For $k > 0$:

$$kx - \omega t + \phi = \text{constant} \Rightarrow \frac{d}{dt}(kx - \omega t) = 0 \Rightarrow k \frac{dx}{dt} - \omega = 0 \Rightarrow \frac{dx}{dt} = \frac{\omega}{k} = +v \quad \text{to } +x$$

$$kx + \omega t + \phi = \text{constant} \Rightarrow \frac{d}{dt}(kx + \omega t) = 0 \Rightarrow k \frac{dx}{dt} + \omega = 0 \Rightarrow \frac{dx}{dt} = -\frac{\omega}{k} = -v \quad \text{to } -x$$

✓ Thus $v = \frac{\omega}{k}$ is the wave (pattern and energy) propagation speed.

4. Snapshot of a traveling wave (at fixed time t_0)



Amplitude A (equilibrium to crest); wavelength λ (crest to crest).

5. Energy transport (qualified statement)

- Traveling waves transfer energy and information.
- In the ideal linear model, material elements oscillate about fixed equilibrium positions, so there is no net bulk transport of the medium.

i Real nonlinear waves can produce small drift or streaming (for example Stokes drift or acoustic streaming).



7. Check your understanding

For $y(x, t) = A \sin(kx - \omega t)$, at fixed $x = x_0$, what is the motion of the medium element?

Answer: simple harmonic motion,
 $y(t) = A \sin(kx_0 - \omega t)$.



8. Think about it

After one period T , how far has the $+x$ traveling pattern moved?

Answer: $vT = \lambda$.



9. Common misconception

The propagation speed v is not generally the instantaneous velocity $\partial y / \partial t$ of a material element.



10. Key takeaway

- Traveling waves carry energy and information through the medium.
- Material elements usually oscillate locally about equilibrium positions.
- The direction and speed of the wave follow from the wave equation.



Today's goal: Understand the form and behavior of traveling waves and how they transport energy.



Next: Wave Speed



Wave Speed in Different Media: What Sets v ?

The propagation speed follows from the medium's restoring forces and inertia.



Key idea: Wave speed is determined by the physical properties and state of the medium.
Frequency-independent speed is a nondispersive approximation, not a universal property of all linear waves.

1 Conceptual Balance

Restoring effect
stronger
→ larger v

Inertia larger
→ smaller v



Wave speed increases with stronger restoring effect and decreases with greater inertia.

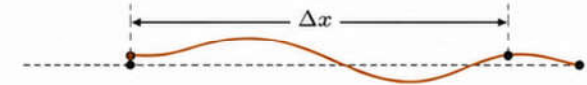
2 What Determines v in Common Media

Wave Type	Restoring Property	Inertia Property	Speed v
Transverse wave on a stretched string	tension T_{tension} (N)	linear mass density μ (kg/m)	$v = \sqrt{\frac{T_{\text{tension}}}{\mu}}$
Longitudinal sound wave in a fluid	bulk modulus B (Pa)	mass density ρ (kg/m ³)	$v = \sqrt{\frac{B}{\rho}}$
Shallow-water gravity wave	gravity g (m/s ²)	water depth h (m)	$v = \sqrt{gh}$

These formulas apply under their stated model assumptions.

3 Derivation for a Stretched String

Consider a small element of string of length Δx .



For small transverse displacements and small slopes,

$$T_{\text{tension}} \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2} \quad \text{Compare with} \quad \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Therefore $v = \sqrt{\frac{T_{\text{tension}}}{\mu}}$

4 Worked Example (String)

$$T_{\text{tension}} = 120 \text{ N}, \quad \mu = 0.020 \text{ kg/m}$$

$$\begin{aligned} v &= \sqrt{\frac{120 \text{ N}}{0.020 \text{ kg/m}}} \\ &= \sqrt{6000 \text{ m}^2/\text{s}^2} \\ &= 77.5 \text{ m/s} \end{aligned}$$

5 Interpretation

- Increasing T_{tension} by a factor of 4 increases v by a factor of 2.
- Increasing μ by a factor of 4 decreases v by a factor of 2.
- At a boundary the source frequency remains continuous; wavelength adjusts through $\lambda = v/f$.

6 Dispersion Qualification

In nondispersive regimes, v is approximately independent of f and amplitude for small-amplitude linear waves.
In dispersive media, phase speed depends on frequency or wave number.

7 Common Pitfall



Wave propagation speed v is not the transverse particle speed $\frac{\partial y}{\partial t}$.



Check Your Understanding

A wave has $f = 40 \text{ Hz}$ and $\lambda = 0.75 \text{ m}$. Find v .

$$v = 30 \text{ m/s}$$



Think About It

If string tension is quadrupled at fixed μ , how does v change?

It doubles.



Key Takeaway

Wave speed emerges from restoring properties divided by inertia.



Model Limits

Use each speed formula only when its assumptions hold: small amplitudes, appropriate geometry, and the stated medium model.



Goal: Use physical insight and derivation to determine wave speeds in common media.



Next: Wave Equation



Worked Example 2: Interpreting a Traveling Wave Equation

Read the coefficients, calculate the wave parameters, and construct a correct snapshot.



Goal: Given a wave function, identify the amplitude, wavelength, frequency, angular frequency, wave number, wave speed, phase, and direction of propagation. Then write the wave after a time Δt .

1. The Problem

A transverse wave on a string is described by

$$y(x, t) = 0.050 \sin(6.00x - 40.0t + \frac{\pi}{4}) \quad (\text{SI units})$$

Determine:

- 1 Amplitude A
- 2 Wave number k and wavelength λ
- 3 Angular frequency ω and frequency f
- 4 Wave speed v
- 5 Direction of propagation
- 6 Phase angle ϕ
- 7 The spatial phase shift relative to $\phi = 0$
- 8 The wave equation at time $t_1 = 0.200$ s

2. Match to the Standard Form

The general form of a traveling wave in the $+x$ direction is

$$y(x, t) = A \sin(kx - \omega t + \phi)$$

Comparing term by term:

Amplitude A	$\rightarrow$	$A = 0.050$ m
Wave number k	$\rightarrow$	$k = 6.00$ rad/m
Angular frequency ω	$\rightarrow$	$\omega = 40.0$ rad/s
Phase angle ϕ	$\rightarrow$	$\phi = \frac{\pi}{4}$ rad = 0.785 rad = 45°

Because the phase has the form $kx - \omega t + \phi$, the wave travels in the $+x$ direction (rightward).

3. Step-by-Step Calculations

- 1 **Wavelength λ**
 $\lambda = \frac{2\pi}{k} = \frac{2\pi}{6.00 \text{ rad/m}} = 1.047 \text{ m}$
- 2 **Frequency f**
 $f = \frac{\omega}{2\pi} = \frac{40.0 \text{ rad/s}}{2\pi} = 6.37 \text{ Hz}$
- 3 **Wave speed v**
 $v = \frac{\omega}{k} = \frac{40.0 \text{ rad/s}}{6.00 \text{ rad/m}} = 6.67 \text{ m/s}$
Check: $v = f\lambda = (6.37 \text{ Hz})(1.047 \text{ m}) = 6.67 \text{ m/s}$ ✓
- 4 **Spatial phase shift relative to $\phi = 0$ at the same time**
 $\Delta x = -\frac{\phi}{k} = -\frac{\pi/4}{6.00} = -\frac{\pi}{24} \text{ m} \approx -0.131 \text{ m}$
Positive ϕ shifts the spatial profile 0.131 m to the left.

4. Wave Equation at $t_1 = 0.200$ s

Substitute $t = 0.200$ s into $y(x, t)$:

$$\begin{aligned} y(x, 0.200) &= 0.050 \sin(6.00x - 40.0(0.200) + \frac{\pi}{4}) \\ &= 0.050 \sin(6.00x - 8.00 + \frac{\pi}{4}) \\ &= 0.050 \sin(6.00x - 7.215) \end{aligned}$$

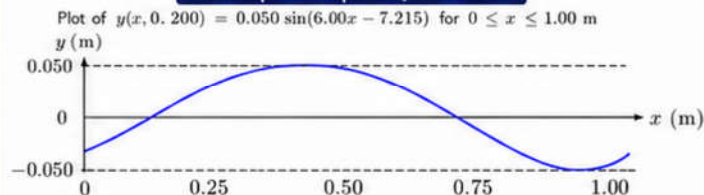
$$y(x, 0.200) = 0.050 \sin(6.00x - 7.215)$$

This is a fixed-time snapshot of the wave, not the history at a fixed position.

5. Physical Interpretation

- Amplitude $A = 0.050$ m: maximum transverse displacement.
- Wavelength $\lambda = 1.047$ m: spatial repeat distance.
- Frequency $f = 6.37$ Hz: 6.37 cycles per second.
- Wave speed $v = 6.67$ m/s: pattern moves to the $+x$ direction.
- At a fixed position x , the string element oscillates up and down in simple harmonic motion while the overall wave pattern moves to the right.
- The wave equation specifies the kinematic description of the wave, not the material law that determines the speed.

6. Snapshot Graph at $t_1 = 0.200$ s



- Key points: $y(0) = -0.040$ m (rising); zero at $x \approx 0.155$ m; maximum $+0.050$ m at $x \approx 0.417$ m; zero at $x \approx 0.679$ m; minimum -0.050 m at $x \approx 0.941$ m; $y(1.00) = -0.047$ m (rising).

7. Summary of Results

Quantity	Symbol	Value	Units	Meaning
Amplitude	A	0.050	m	Maximum displacement
Wavelength	λ	1.047	m	Distance between crests
Wave number	k	6.00	rad/m	Spatial frequency
Angular frequency	ω	40.0	rad/s	Temporal frequency (rad/s)
Frequency	f	6.37	Hz	Cycles per second
Wave speed	v	6.67	m/s	Speed of wave
Phase angle	ϕ	$\frac{\pi}{4} = 0.785$	rad	Initial phase
Direction	—	$+x$ (right)	—	Direction of propagation



Check Your Understanding

Given $y(x, t) = 0.030 \sin(4.00x + 20.0t - \frac{\pi}{6})$.
Find A , λ , f , v , ϕ and the direction.

$A = 0.030$ m
 $\lambda = \frac{2\pi}{4.00} = \frac{\pi}{2} \approx 1.571$ m
 $f = \frac{20.0}{2\pi} = \frac{10}{\pi} \approx 3.18$ Hz
 $v = \frac{\omega}{k} = \frac{20.0}{4.00} = 5.00$ m/s
 $\phi = -\frac{\pi}{6}$ rad = -30°
Direction: $-x$ (leftward) since $kx + \omega t$



Think About It

If the wave speed remains constant and the frequency doubles, what happens to the wavelength?

The wavelength halves.

$$(\lambda = \frac{v}{f})$$



Common Pitfall

For $k > 0$, $kx - \omega t + \phi$ travels in $+x$ (rightward), whereas $kx + \omega t + \phi$ travels in $-x$ (leftward).
The sign in front of ωt tells the direction!



Today's goal: Learn how to extract physical meaning from a wave equation and apply the concepts to solve problems.



Next: Standing Waves



Standing Waves

Opposite traveling waves create a stationary interference pattern.



Key idea: A standing wave is formed by two coherent waves of equal frequency, wavelength, and amplitude traveling in opposite directions. Nodes remain at zero displacement; antinodes oscillate with maximum amplitude.

1. Formation from Two Waves

Two traveling waves of equal amplitude A and frequency:

$$y_1(x, t) = A \sin(kx - \omega t) \quad \rightarrow$$

$$y_2(x, t) = A \sin(kx + \omega t) \quad \leftarrow$$

Superposition (standing wave):

$$y(x, t) = y_1 + y_2 = 2A \sin(kx) \cos(\omega t)$$

- $2A|\sin(kx)|$ is the position-dependent oscillation amplitude.
- $\cos(\omega t)$ gives the common time oscillation.
- The envelope is fixed in space; the string is not frozen.

2. Nodes and Antinodes

Nodes (zero displacement):

$$\sin(kx) = 0$$

$$kx = n\pi$$

$$x_N = \frac{n\pi}{k} = \frac{n\lambda}{2}, \quad n = 0, 1, 2, \dots$$

Antinodes (maximum amplitude):

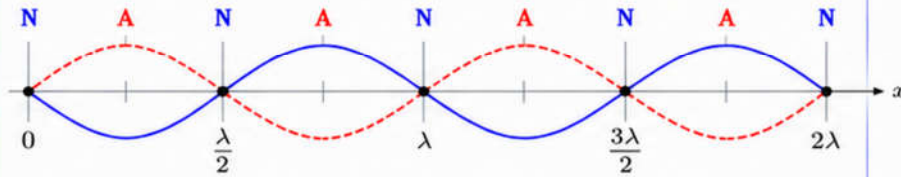
$$|\sin(kx)| = 1$$

$$kx = \frac{(2n+1)\pi}{2}$$

$$x_A = \frac{(2n+1)\pi}{2k} = \frac{(2n+1)\lambda}{4}, \quad n = 0, 1, 2, \dots$$

3. Node/Antinode Pattern (0 to 2λ)

— Instantaneous shape at $t = 0$ - - - Instantaneous shape at $t = \frac{\pi}{\omega}$



Nodes at $x_N = n\lambda/2$

Antinodes at $x_A = (2n+1)\lambda/4$

Amplitude: If each traveling wave has amplitude A , the maximum standing-wave amplitude at an antinode is $2A$.

4. Fixed-Fixed String: Normal Modes (Introduction)

For a string of length L fixed at both ends:

$$y(0, t) = y(L, t) = 0$$

Allowed wavelengths: $\lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3, \dots)$

Wave numbers: $k_n = \frac{n\pi}{L}$

Resonant frequencies: $f_n = \frac{nv}{2L} \quad (n = 1, 2, 3, \dots)$

$n = 1$ (Fundamental)

$$\lambda_1 = 2L$$

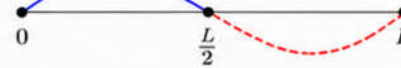


Nodes: 0, L

Antinode: $L/2$

$n = 2$ (Second harmonic)

$$\lambda_2 = L$$



Nodes: 0, $L/2$, L

Antinodes: $L/4$, $3L/4$

$n = 3$ (Third harmonic)

$$\lambda_3 = \frac{2L}{3}$$



Nodes: 0, $L/3$, $2L/3$, L

Antinodes: $L/6$, $L/2$, $5L/6$



5. Check

$L = 0.80 \text{ m}$, $v = 200 \text{ m/s}$,
fixed at both ends.
 $f_1 = 125 \text{ Hz}$, $f_2 = 250 \text{ Hz}$, $f_3 = 375 \text{ Hz}$.



6. Think

If L increases while v remains constant, resonant frequencies decrease inversely with L .



7. Common Pitfall

Nodes are points of zero displacement amplitude, not points where the string ceases to exist; material between nodes oscillates.



8. Takeaway

Boundary conditions select discrete standing-wave patterns.



Today's goal: Understand how standing waves form, how nodes and antinodes arise, and how boundary conditions lead to discrete normal modes.



Next: Boundary Conditions



Boundary Conditions and Allowed Modes

The endpoints determine where nodes and antinodes occur — and which wavelengths can resonate.

Notation

N = node, zero displacement amplitude
 A = antinode, maximum displacement amplitude
 L = string length
Mode index $n = 1, 2, 3, \dots$

LEFT: FIXED–FIXED STRING (both ends fixed)

Boundary conditions:

$$y(0, t) = 0 \quad \text{and} \quad y(L, t) = 0$$

Both endpoints are nodes.

Allowed modes:

$$y_n(x) \propto \sin\left(\frac{n\pi x}{L}\right)$$

$$k_n = \frac{n\pi}{L} \quad \lambda_n = \frac{2L}{n}$$

$$f_n = \frac{nv}{2L}, \quad n = 1, 2, 3, \dots$$

Mode	Instantaneous mode shape $y(x)$	Nodes (N)	Antinodes (A)
$n = 1$		0, L	$L/2$
$n = 2$		0, $L/2$, L	$L/4$, $3L/4$
$n = 3$		0, $L/3$, $2L/3$, L	$L/6$, $L/2$, $5L/6$

RIGHT: FIXED–FREE STRING (one end fixed, one end free)

Boundary conditions:

$$y(0, t) = 0 \quad \left. \frac{\partial y}{\partial x} \right|_{x=L} = 0$$

The fixed endpoint is a node;
the ideal free endpoint is an antinode
with horizontal tangent.

Allowed modes:

$$y_n(x) \propto \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

$$k_n = \frac{(2n-1)\pi}{2L} \quad \lambda_n = \frac{4L}{2n-1}$$

$$f_n = \frac{(2n-1)v}{4L}, \quad n = 1, 2, 3, \dots$$

Only odd (quarter-wave) harmonics occur: $f_1, 3f_1, 5f_1, \dots$

Mode	Instantaneous mode shape $y(x)$	Nodes (N)	Antinodes (A)
$n = 1$		0	L
$n = 2$		0, $2L/3$	$L/3$, L
$n = 3$		0, $2L/5$, $4L/5$	$L/5$, $3L/5$, L



Comparison

- Fixed–fixed: both ends nodes; all positive integer harmonics $n = 1, 2, 3, \dots$
- Fixed–free: node at fixed end and antinode at free end; only odd harmonics $n = 1, 3, 5, \dots$
- Boundary conditions quantize k , λ and f .



Why must the free end have zero slope?

Because the transverse component of the endpoint tension vanishes when there is no external transverse force.

$$F_y = T \frac{\partial y}{\partial x} = 0 \Rightarrow \left. \frac{\partial y}{\partial x} \right|_{x=L} = 0$$



Check Example

$L = 1.20 \text{ m}$, $v = 240 \text{ m/s}$,
fixed at one end and free at the other.
 $f_1 = 50 \text{ Hz}$, $f_2 = 150 \text{ Hz}$, $f_3 = 250 \text{ Hz}$.

For $n = 2$ (second mode):

- Nodes at 0 and 0.80 m ($= 2L/3$).
- Antinodes at 0.40 m ($= L/3$) and 1.20 m ($= L$).



Common Pitfall

- A free end is not a displacement node; in the ideal string model it is a displacement antinode.



Today's goal: Understand how boundary conditions shape standing waves and determine the allowed modes.



Next: Worked Example 3



Worked Example 3: Standing Waves on a String – Resonant Frequencies

Apply the fixed-fixed boundary conditions to obtain discrete wavelengths and frequencies.



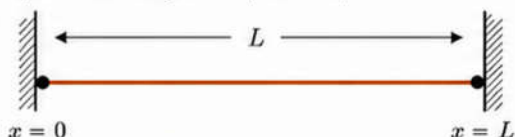
Problem: A string of length $L = 1.20$ m is fixed at both ends.
Wave speed $v = 240$ m/s. Find the first three resonant frequencies f_1, f_2, f_3
and the corresponding wavelengths $\lambda_1, \lambda_2, \lambda_3$.

Boundary Conditions (fixed-fixed)

$$y(0, t) = 0 \quad \text{and} \quad y(L, t) = 0$$

1 Physical Situation

A string of length L fixed at both ends.
Only certain wavelengths “fit” and
produce standing waves (resonance).



2 Key Relations (Fixed-Fixed String)

- Allowed wavelengths: $\lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3, \dots)$
- Allowed wave numbers: $k_n = \frac{n\pi}{L}$
- Resonant frequencies: $f_n = \frac{nv}{2L} \quad (n = 1, 2, 3, \dots)$
where v is the wave speed on the string.

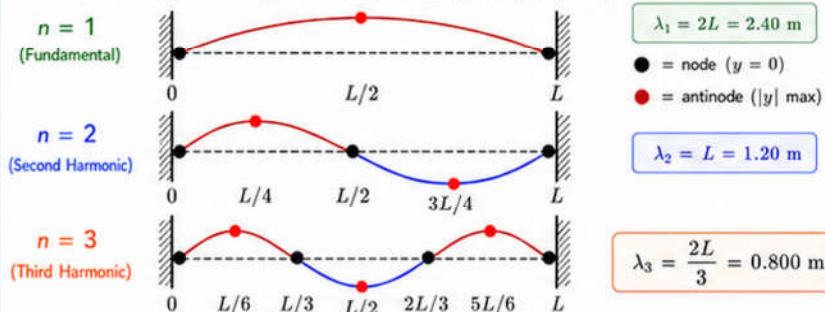
3 Step-by-Step Calculation of Frequencies

Given: $L = 1.20$ m, $v = 240$ m/s

- $n = 1$: $\lambda_1 = \frac{2(1.20)}{1} = 2.40$ m $f_1 = \frac{1(240)}{2(1.20)} = \frac{240}{2.40} = 100$ Hz
- $n = 2$: $\lambda_2 = \frac{2(1.20)}{2} = 1.20$ m $f_2 = \frac{2(240)}{2(1.20)} = \frac{480}{2.40} = 200$ Hz
- $n = 3$: $\lambda_3 = \frac{2(1.20)}{3} = 0.800$ m $f_3 = \frac{3(240)}{2(1.20)} = \frac{720}{2.40} = 300$ Hz

4 Mode Shapes (Fixed-Fixed String)

The n th mode has n antinodes and $n + 1$ nodes (including endpoints).



Black dots: nodes ($y = 0$) | Red dots: antinodes (maximum displacement)

5 Results Summary

n	mode	λ_n (m)	f_n (Hz)	nodes (positions)	antinodes (positions)
1	Fundamental	2.40	100	0, L	$L/2$
2	Second Harmonic	1.20	200	0, $L/2$, L	$L/4$, $3L/4$
3	Third Harmonic	0.800	300	0, $L/3$, $2L/3$, L	$L/6$, $L/2$, $5L/6$

- n th mode has n antinodes and $n + 1$ nodes (including endpoints).
- $f_n = nf_1$ for an ideal, nondispersive string.
- Higher $n \rightarrow$ shorter wavelength and higher frequency.
- Frequency spacing $\Delta f = f_1 = 100$ Hz.

6 Think, Check & Common Pitfall



Think About It

If the length L were doubled while the wave speed v stays the same, what happens to the resonant frequencies?



Check Your Understanding

A string of length $L = 0.90$ m has wave speed $v = 180$ m/s. What are f_1, f_2 , and f_3 ?



Common Pitfall

Do not omit the factor of 2 in $\lambda_n = \frac{2L}{n}$ and $f_n = \frac{nv}{2L}$.
It comes from the fixed-fixed boundary!



Key Takeaway: Boundary conditions turn a continuous range of wavelengths into discrete normal modes.



Next: Engineering Applications



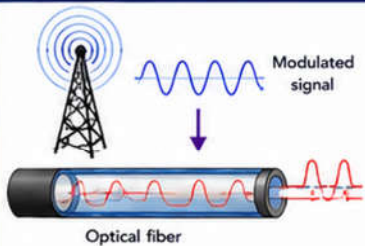
Engineering Applications of Waves

Wave principles enable communication, imaging, sensing, safer structures, and energy transfer.



Opening idea: Engineers control wave generation, propagation, reflection, interference, resonance, attenuation, and detection to extract information or deliver energy.

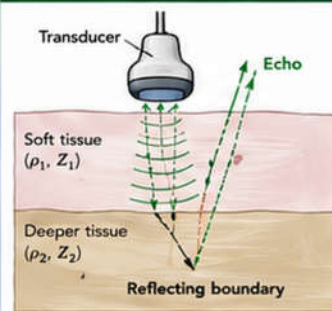
1. COMMUNICATIONS AND PHOTONICS



- Information is encoded by modulation of amplitude, frequency, phase, or pulses.
- Examples: radio, microwave links, optical-fiber communication.
- Wavelength and medium determine propagation, attenuation, and device size.

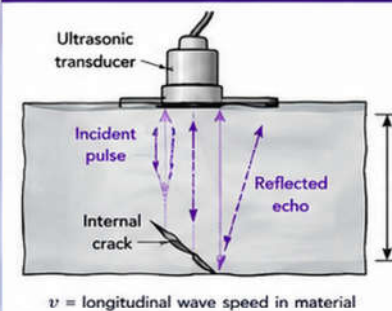
Relation: $\lambda = \frac{v}{f}$ (wavelength = speed / frequency)

2. MEDICAL IMAGING AND THERAPY



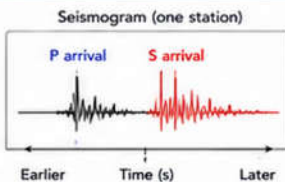
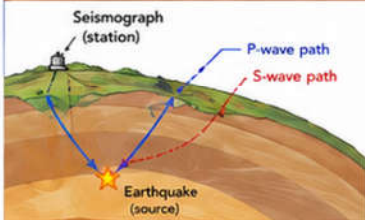
- Ultrasound forms images from echo time and amplitude.
- Pulse-echo depth:
$$d = \frac{v_{\text{tissue}} \Delta t}{2}$$
- Acoustic impedance mismatch controls reflection.
- MRI uses RF electromagnetic excitation and magnetic-field gradients (not ultrasound).
- Examples: diagnostic ultrasound, focused ultrasound therapy.

3. NON-DESTRUCTIVE TESTING AND INDUSTRIAL SENSING



- Echo time locates a flaw:
$$d = \frac{v \Delta t}{2}$$
- Echo strength depends on defect geometry and impedance contrast.
- Examples: weld inspection, thickness measurement, guided-wave monitoring.

4. SEISMOLOGY AND EARTH MONITORING



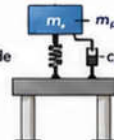
- P waves usually travel faster than S waves in solids.
- Arrival-time differences help locate earthquakes.
- Reflection and refraction reveal subsurface structure.
- S waves do not propagate through fluids.

5. STRUCTURAL DYNAMICS, VIBRATION AND ACOUSTICS

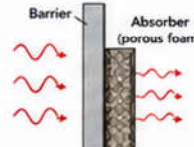
Structural mode shape (first mode)



Tuned mass damper



Acoustic control



- Modal analysis identifies natural frequencies and mode shapes.
- Excitation near resonance can amplify vibration.
- Damping, isolation and tuned absorbers reduce response.
- Acoustic absorption, barriers and active control reduce noise.



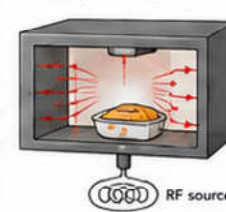
Avoid sustained forcing near f_n unless resonance is intended.

6. WAVE-BASED ENERGY AND PROCESSING

Ocean wave energy converter



Microwave/RF heating and power transfer



Ultrasonic processing (ultrasonic cleaning)



- Ocean surface-wave energy can be converted to electrical power.
- Electromagnetic waves deliver energy in microwave/RF heating and wireless power transfer.
- High-intensity ultrasound supports cleaning, welding, mixing, and sonochemistry.



Spotlight Example (Ultrasound Imaging)

Wave type: Mechanical (sound)
Source: Piezoelectric transducer pulse
Medium: Body tissues

Detector: Same transducer (echo)
Measured quantity: Echo time and amplitude
Key phenomenon: Reflection; impedance mismatch

Depth: $d = \frac{v \Delta t}{2}$
(round trip)



Common Pitfall

Do not confuse wave propagation speed with material-particle speed, and do not apply a medium-specific formula outside its assumptions.



Today's Goal

Understand how wave principles are applied across engineering to communicate, image, sense, protect structures, and transfer or process energy.



Across applications, the same questions recur: What sets v ? What reflects or transmits? Where does resonance occur? How is energy attenuated or detected?



Next: Common Mistakes

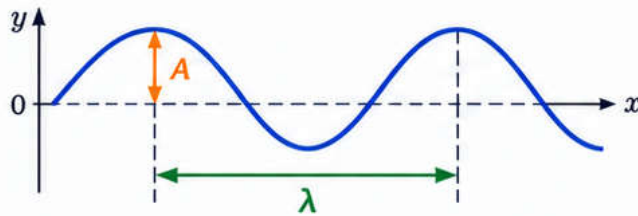
Common Mistakes — and the Quick Fix



Check definitions, units, direction, phase, and boundary conditions.

1 Amplitude and wavelength

A: maximum absolute displacement from equilibrium.
 λ : shortest positive spacing between equal-phase points.



A is a displacement; λ is a spatial period.

2 Keep the 2π factors

$$\begin{array}{ll} f = 1/T & \text{Hz} = \text{s}^{-1} \\ \omega = 2\pi f & \text{rad s}^{-1} \\ k = 2\pi/\lambda & \text{rad m}^{-1} \end{array}$$

Spatial frequency in cycles/m: $1/\lambda = k/(2\pi)$.

k is the angular wavenumber.

3 Read the propagation sign

For $k > 0$ and $\omega > 0$:

$kx - \omega t \rightarrow$ propagation in $+x$

$kx + \omega t \rightarrow$ propagation in $-x$

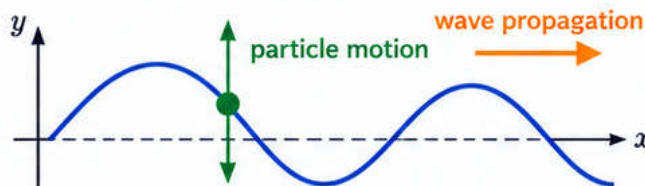
Follow a point of constant phase:

$$kx - \omega t = C \Rightarrow \frac{dx}{dt} = +\frac{\omega}{k}$$

$\xrightarrow{+x}$

4 Wave motion and particle motion

A traveling disturbance can transport energy.
 Medium particles oscillate about equilibrium.



Transverse wave

Wave speed is not particle velocity.

5 Retain the phase constant

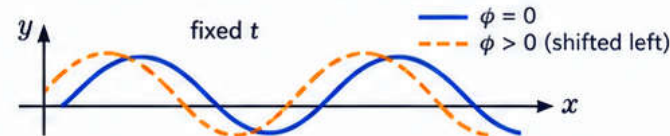
$$y(x, t) = A \sin(kx - \omega t + \phi)$$

Relative to $\phi = 0$, with the same origins:

Snapshot shift: $\Delta x = -\phi/k$

Time shift at fixed x : $\Delta t = +\phi/\omega$

ϕ follows the given data and the chosen origins.



6 Use the correct boundaries

Normal modes of an ideal uniform string:

Fixed end: $y = 0$ (node)

Free end: $\partial y / \partial x = 0$ (antinode)

Fixed-fixed: $\lambda_n = 2L/n$

Fixed-free: $\lambda_n = 4L/(2n - 1)$

$n = 1, 2, 3, \dots$

Match the boundary condition to the physical variable.



Final check: Definitions • units • direction • phase • boundaries • model assumptions

Summary: Waves and Oscillations



Oscillation in time, variation in space, and a propagation law.

Oscillation in time

$$x(t) = A \cos(\omega t + \phi_0)$$

$$f = 1/T$$

$$\omega = 2\pi f$$

Variation in space

λ : spatial period

$$k = 2\pi/\lambda$$

k in rad m^{-1}

Phase speed

$$v_p = f\lambda = \omega/k$$

Speed of a constant-phase point.

Traveling wave

$$y(x, t) = A \sin(kx - \omega t + \phi)$$

For $k > 0$ and $\omega > 0$:

$$kx - \omega t \rightarrow +x$$

$$kx + \omega t \rightarrow -x$$

1 Traveling waves

- The phase pattern propagates.
- Medium particles oscillate locally.
- A disturbance can transport energy.
- Wave speed differs from particle velocity.

Phase speed need not equal energy or signal speed.

Direction check: follow a constant phase.

$$kx - \omega t = \text{constant}$$

$$\Rightarrow dx/dt = +\omega/k$$

Here x is the location of the phase point,
not a material particle.

2 Standing waves

Equal-amplitude waves of the same frequency travel in opposite directions.

- **Nodes:** zero oscillation amplitude.
- **Antinodes:** maximum oscillation amplitude.
- **Boundaries** select the allowed modes.

Ideal uniform, nondispersive string:

$$\text{Fixed-fixed: } f_n = nv_p/(2L)$$

$$\text{Fixed-free: } f_n = (2n - 1)v_p/(4L)$$

$n = 1, 2, 3, \dots$; v_p is constant.

3 Medium and model

- Mechanical waves require a medium.
- Their speed depends on inertia and restoring forces.
- Electromagnetic waves can travel in vacuum.
- Interfaces may produce reflection and refraction.

Before choosing a formula, check:

Geometry • boundary conditions • dispersion
Units • sign convention • assumptions

1 Identify

A, k, ω, ϕ and direction



2 Convert

λ, f, T and v_p



3 Apply

Medium and boundaries



4 Verify

Units, direction and limits

Independent data and the correct physical model determine the remaining quantities.