
PHYSICS FOR ENGINEERING STUDENTS

Circular and rotational motion

Definitions to applications

Lecture 4 — Book-style student notes

Learning focus

Angular position, angular position

Uniform circular motion

Canonical pendulum

Moment of Inertia

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Circular and Rotational Motion

Circular and rotational motion extend the familiar ideas of position, velocity, acceleration, force, energy, and momentum to systems that turn about an axis. The mathematical structure closely mirrors translational mechanics, but rotation introduces new physical quantities: angular position, angular velocity, angular acceleration, torque, moment of inertia, and angular momentum. These tools are essential for analysing wheels, turbines, motors, robotic joints, spacecraft, and countless other engineering systems.

Learning objectives

- Describe circular motion using angular position, angular velocity, and angular acceleration.
- Relate angular quantities to tangential and radial motion at any radius.
- Apply centripetal-force equations to curves, pendulums, and orbits.
- Calculate torque and use the rotational form of Newton's second law.
- Determine moments of inertia and apply the parallel-axis theorem.
- Use rotational work, energy, impulse, and angular-momentum conservation.
- Analyse coupled translation and rotation, including a massive-pulley Atwood machine.

1. Translational and rotational quantities

The correspondence between translation and rotation is one of the most useful organising ideas in mechanics. Linear position x corresponds to angular position θ ; velocity v corresponds to angular velocity ω ; acceleration a corresponds to angular acceleration α ; force F corresponds to torque τ ; mass m corresponds to moment of inertia I ; and linear momentum p corresponds to angular momentum L .

Translation	Rotation	Meaning
x	θ	Position
$v = dx/dt$	$\omega = d\theta/dt$	Rate of change of position
$a = dv/dt$	$\alpha = d\omega/dt$	Rate of change of velocity
F	$\tau = \mathbf{r} \times \mathbf{F}$	Cause of acceleration
m	I	Resistance to acceleration
$p = mv$	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$ or $I\omega$	Quantity of motion
$K = \frac{1}{2}mv^2$	$K_{\text{rot}} = \frac{1}{2}I\omega^2$	Kinetic energy

The analogy is powerful but must not be used blindly. A force and a torque have different units and different physical meanings. Likewise, mass depends only on how much matter is present, whereas moment of inertia also depends on how that mass is distributed relative to the chosen axis.

Radians and arc length

Angular relations take their simplest form when angles are expressed in radians. For a circle of radius r , an angular displacement $\Delta\theta$ subtends arc length s :

$$s = r\Delta\theta.$$

One complete revolution is 2π radians = 360° . Because the radian is defined as a ratio of two lengths, it is dimensionless, although the symbol rad is retained to distinguish angular quantities.

2. Angular position and angular displacement

Angular position θ locates a point on a circle relative to a reference line. The angular displacement between initial and final states is

$$\Delta\theta = \theta_f - \theta_i.$$

By convention, counterclockwise rotation is often positive and clockwise rotation negative. The sign convention is arbitrary, but it must remain consistent. Angular positions are not restricted to the interval from 0 to 2π . A point that makes several turns accumulates an angle larger than 2π , allowing the total rotation to be tracked.

Examples

If a wheel moves counterclockwise from 30° to 120° , $\Delta\theta = +90^\circ = +\pi/2$ rad. If it moves clockwise from 120° to 30° , $\Delta\theta = -90^\circ = -\pi/2$ rad. A total of 1.5 counterclockwise revolutions corresponds to $\Delta\theta = 1.5(2\pi) = 3\pi$ rad.

For a radius of 0.25 m and an arc length of 0.50 m, the angular displacement is $\Delta\theta = s/r = 2.0$ rad. The same angular displacement applies to every point of a rigid body but points farther from the axis travel greater arc lengths.

Key distinction: Angular displacement describes rotation of the whole rigid body. Linear displacement along the circular path depends on radius through $s = r\Delta\theta$.

3. Angular velocity and angular acceleration

Angular velocity

Average angular velocity is angular displacement divided by elapsed time, while instantaneous angular velocity is its limiting value:

$$\omega_{\text{avg}} = \Delta\theta/\Delta t, \quad \omega = d\theta/dt.$$

Angular velocity is a vector along the axis of rotation. Its direction follows the right-hand rule: curl the fingers of the right hand in the direction of rotation, and the thumb points along $\vec{\omega}$. For rotation in a fixed plane, the vector direction may be represented by a sign.

If angular velocity is constant, angular position changes linearly:

$$\theta(t) = \theta_0 + \omega t.$$

Useful conversions are 1 revolution per second = 2π rad/s and 1 rpm = $2\pi/60$ rad/s. Thus 1800 rpm corresponds to 188.5 rad/s.

Angular acceleration

Angular acceleration measures the rate at which angular velocity changes:

$$\alpha_{\text{avg}} = \Delta\omega/\Delta t, \quad \alpha = d\omega/dt = d^2\theta/dt^2.$$

The sign of α does not by itself indicate speeding up or slowing down. The magnitude of ω increases when ω and α have the same sign and decreases when they have opposite signs. For example, $\omega < 0$ and $\alpha < 0$ means increasingly rapid clockwise rotation.

Constant-angular-acceleration equations

$$\omega = \omega_0 + \alpha t,$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2,$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0),$$

$$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t.$$

These equations are exact only when α is constant. They are the rotational counterparts of constant-acceleration kinematics and require angles in radians.

Worked example: braking a flywheel

A flywheel slows uniformly from 100 rad/s to 20 rad/s with $\alpha = -5.0$ rad/s². From $\omega = \omega_0 + \alpha t$, the time is $t = (20 - 100)/(-5.0) = 16$ s. The angular displacement during braking is $\theta - \theta_0 = \frac{1}{2}(100 + 20)(16) = 960$ rad, or approximately 153 revolutions. A large number of turns during braking is plausible because the initial angular speed is high.

4. Relations between linear and angular motion

For a rigid body rotating about a fixed axis, all points share the same θ , ω , and α . Linear quantities depend on distance r from the axis. Differentiating $s = r\theta$ gives

$$v_t = r\omega, \quad a_t = r\alpha.$$

Tangential velocity is tangent to the circular path. Tangential acceleration changes the speed. In addition, curved motion requires radial or centripetal acceleration directed toward the centre:

$$a_r = v^2/r = r\omega^2.$$

The tangential and radial components are perpendicular. Therefore, when both are present,

$$a = \sqrt{(a_t^2 + a_r^2)}.$$

Vector directions at a point

The position vector $\mathbf{r}$ points outward from the axis to the point. The velocity $\mathbf{v}$ is tangent in the direction of motion. The radial acceleration points inward, opposite $\mathbf{r}$. Tangential acceleration points along $\mathbf{v}$ when speed increases and opposite $\mathbf{v}$ when speed decreases. Mixing these directions is a frequent source of errors.

Worked example

A point lies 0.40 m from the axis of a rotor with $\omega = 12 \text{ rad/s}$ and $\alpha = 3.0 \text{ rad/s}^2$. Its tangential speed is $v = r\omega = 4.8 \text{ m/s}$. Its tangential acceleration is $a_t = r\alpha = 1.2 \text{ m/s}^2$, while its radial acceleration is $a_r = r\omega^2 = 57.6 \text{ m/s}^2$. The total acceleration is $\sqrt{(1.2^2 + 57.6^2)} \approx 57.6 \text{ m/s}^2$, almost entirely radial.

5. Uniform circular motion

In uniform circular motion, speed and angular speed are constant. Angular acceleration and tangential acceleration are zero, but radial acceleration is non-zero because velocity direction changes continuously. The kinematic relations are

$$\theta(t) = \theta_0 + \omega t, \quad v = r\omega, \quad a_r = v^2/r = r\omega^2.$$

Period and frequency

The period T is the time for one revolution, and frequency f is the number of revolutions per unit time:

$$f = 1/T, \quad \omega = 2\pi f = 2\pi/T, \quad v = 2\pi r/T.$$

A constant speed does not mean zero acceleration. Since acceleration is the rate of change of the velocity vector, a continuous change in direction is sufficient to produce acceleration.

Satellite example

A satellite moves at 7.5 km/s in a circular orbit of radius 7000 km. Converting to SI units gives $v = 7.5 \times 10^3 \text{ m/s}$ and $r = 7.0 \times 10^6 \text{ m}$. Its centripetal acceleration is

$$a_r = v^2/r = (7.5 \times 10^3)^2 / (7.0 \times 10^6) = 8.04 \text{ m/s}^2.$$

For a 500 kg satellite, the required inward force is $4.02 \times 10^3 \text{ N}$. Gravity supplies this force. The orbital period is $T = 2\pi r/v \approx 5.86 \times 10^3 \text{ s} = 97.7 \text{ min}$.

6. Centripetal force and engineering applications

Centripetal force is not an additional physical force. It is the name for the net radial component of the real forces acting toward the centre:

$$\Sigma F_{\text{inward}} = ma_r = mv^2/r = m\omega^2 r.$$

Tension, gravity, friction, normal force, or a combination of these may provide the required radial resultant. An FBD must contain only the real forces; the phrase “centripetal force” belongs in the radial equation, not as an extra arrow.

Conical pendulum

A mass on a string moves in a horizontal circle while the string makes angle θ with the vertical. Tension T has a vertical component balancing weight and a horizontal component providing the radial resultant:

$$T \cos\theta = mg, \quad T \sin\theta = mv^2/r.$$

Dividing gives $\tan\theta = v^2/(rg)$. The string tension is greater than mg because it must supply both vertical support and horizontal acceleration.

Frictionless banked curve

On a banked road of angle θ with no friction, the normal force has vertical and horizontal components:

$$N \cos\theta = mg, \quad N \sin\theta = mv^2/r.$$

$$\tan\theta = v^2/(rg), \quad v = \sqrt{rg \tan\theta}.$$

This is the design speed at which no lateral friction is required. Actual roads rely on friction to allow a range of safe speeds around this ideal value.

Flat curve with static friction

On a level road, $N = mg$ and static friction provides the inward force. The required friction is $f_s = mv^2/R$ and must not exceed $\mu_s N$. Therefore

$$mv^2/R \leq \mu_s mg, \quad v_{\text{max}} = \sqrt{(\mu_s g R)}.$$

Static friction is not automatically at its maximum. Below v_{max} it adjusts to the value required for circular motion. If the speed exceeds v_{max} , the available friction is insufficient and the vehicle initially skids approximately tangent to the curve, appearing to move outward relative to the turning road.

7. Torque: the rotational effect of force

Torque describes the tendency of a force to rotate an object about a chosen point or axis. It is defined by the vector product

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

Its magnitude is

$$\tau = rF \sin\phi = Fd_{\perp},$$

where ϕ is the angle between $\mathbf{r}$ and $\mathbf{F}$, and $d_{\perp}$ is the perpendicular distance from the axis to the force's line of action. Only the force component perpendicular to $\mathbf{r}$ contributes. The SI unit is N.m, which is dimensionally equal to a joule but represents a different physical quantity and should not be called a joule.

Direction and sign

Torque is perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{F}$. Use the right-hand rule: point the fingers along $\mathbf{r}$ and curl them toward $\mathbf{F}$; the thumb gives the torque direction. In planar problems, counterclockwise torque is usually positive and clockwise torque negative.

Door example

A 25 N force is applied perpendicular to a door 0.90 m from the hinge. The torque magnitude is $\tau = (0.90)(25)\sin 90^\circ = 22.5 \text{ N}\cdot\text{m}$. Applying the same force halfway to the hinge produces half the torque. Applying the force toward the hinge produces zero torque because its line of action passes through the axis.

Engineering interpretation: Torque depends not only on force magnitude but also on where and in what direction the force is applied. A longer lever arm can reduce the force needed for the same rotational effect.

8. Rotational dynamics and moment of inertia

Newton's second law for rotation

For rotation about a fixed axis, the net external torque causes angular acceleration:

$$\Sigma \tau_{\text{ext}} = I\alpha.$$

This is the rotational analog of $\Sigma F = ma$. For a fixed torque, increasing moment of inertia decreases angular acceleration. If net torque is zero, $\alpha = 0$, so the body is either at rest or rotates with constant angular velocity.

Definition of moment of inertia

For particles at perpendicular distances r_i from an axis,

$$I = \sum m_i r_i^2.$$

For a continuous body,

$$I = \int r^2 dm.$$

The r^2 dependence makes mass far from the axis especially important. Two bodies with equal mass and radius may have different moments of inertia because their mass distributions differ.

Object and central axis	Moment of inertia
Point mass at radius R	MR^2
Thin hoop or ring	MR^2
Solid disk or cylinder	$\frac{1}{2}MR^2$
Solid sphere	$\frac{2}{5}MR^2$
Thin rod, centre, perpendicular	$(1/12)ML^2$
Thin rod, end, perpendicular	$(1/3)ML^2$

Worked disk example

A solid disk has $M = 2.0$ kg and $R = 0.30$ m. Its central moment of inertia is $I = \frac{1}{2}MR^2 = 0.090$ kg·m². A constant torque of 1.2 N·m produces $\alpha = \tau/I = 13.3$ rad/s². If the disk starts from rest, after 4.0 s its angular velocity is 53.3 rad/s and its angular displacement is $\frac{1}{2}\alpha t^2 = 106.7$ rad.

9. Axis theorems**Parallel-axis theorem**

If I_{cm} is known about an axis through the centre of mass, the moment of inertia about a parallel axis displaced by perpendicular distance d is

$$I = I_{cm} + Md^2.$$

For a thin rod of length L , $I_{cm} = (1/12)ML^2$. Shifting to a parallel axis through one end, $d = L/2$, gives $I = (1/12)ML^2 + M(L/2)^2 = (1/3)ML^2$, matching the standard result.

Perpendicular-axis theorem

For a thin planar lamina lying in the x - y plane, the moment of inertia about the perpendicular z -axis through the same point satisfies

$$I_z = I_x + I_y.$$

This theorem applies only to planar bodies. It must not be used indiscriminately for three-dimensional objects. Both axis theorems require careful identification of the exact axes; a moment of inertia is incomplete information unless its axis is specified.

Example: ring about a tangent axis

A thin ring has $I_{\text{cm}} = MR^2$ about the central axis perpendicular to its plane. A parallel tangent axis is distance R away, so $I = MR^2 + MR^2 = 2MR^2$. The same ring is therefore twice as resistant to angular acceleration about the tangent axis as about the central axis.

10. Rotational work, energy, and power

An incremental rotational displacement $d\theta$ under torque τ transfers work

$$dW = \tau d\theta, \quad W = \int \tau d\theta.$$

For constant torque, $W = \tau\Delta\theta$. The rotational kinetic energy of a rigid body about a fixed axis is

$$K_{\text{rot}} = \frac{1}{2}I\omega^2.$$

The rotational work–energy theorem is

$$W_{\text{net}} = \Delta K_{\text{rot}} = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2.$$

Instantaneous rotational power is

$$P = \tau\omega.$$

The equation mirrors $P = \mathbf{F} \cdot \mathbf{v}$ in translation. A motor that delivers $40 \text{ N}\cdot\text{m}$ at 150 rad/s supplies $6000 \text{ W} = 6.0 \text{ kW}$ of mechanical power.

Rolling bodies

A rolling object generally has both translational and rotational kinetic energy:

$$K = \frac{1}{2}Mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2.$$

For rolling without slipping, the point of contact has zero instantaneous velocity relative to the surface, and

$$v_{\text{cm}} = R\omega, \quad a_{\text{cm}} = R\alpha.$$

For a solid cylinder, $I_{\text{cm}} = \frac{1}{2}MR^2$, so $K = \frac{1}{2}Mv^2 + \frac{1}{4}Mv^2 = \frac{3}{4}Mv^2$. Ignoring rotational energy would underestimate the energy and give an incorrect speed.

11. Angular momentum and angular impulse

For a particle with position $\mathbf{r}$ and linear momentum $\mathbf{p}$ relative to a point, angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

Its magnitude is $L = rp \sin\phi$. For a rigid body rotating about a fixed principal axis,

$$L = I\omega.$$

The direction follows the right-hand rule and depends on the chosen point or axis. The SI unit is $\text{kg}\cdot\text{m}^2/\text{s}$.

Torque changes angular momentum

$$\Sigma \tau_{\text{ext}} = dL/dt.$$

This is the most general rotational form of Newton's second law. When I is constant and the direction of the axis is fixed, it reduces to $\Sigma \tau = I\alpha$. Integrating over time defines angular impulse:

$$\int \Sigma \tau_{\text{ext}} dt = \Delta L.$$

A short braking torque can produce a large change in angular momentum if its magnitude is large; a smaller torque can produce the same change if applied for a longer time.

12. Conservation of angular momentum

If the net external torque on a system is zero, total angular momentum is constant:

$$\Sigma \tau_{\text{ext}} = 0 \Rightarrow L = \text{constant}.$$

For fixed-axis rotation,

$$I_1 \omega_1 = I_2 \omega_2.$$

Angular velocity is not generally conserved. If the mass distribution changes, I changes and ω adjusts so that $I\omega$ remains constant. Internal forces may rearrange the system but cannot change its total angular momentum.

Figure skater

A skater initially has $I_1 = 4.0 \text{ kg}\cdot\text{m}^2$ and $\omega_1 = 2.0 \text{ rad/s}$. She pulls in her arms until $I_2 = 1.6 \text{ kg}\cdot\text{m}^2$. Neglecting external torque, $\omega_2 = I_1 \omega_1 / I_2 = 5.0 \text{ rad/s}$. Her rotational kinetic energy increases because she performs internal work while pulling her arms inward; angular-momentum conservation does not imply energy conservation for this internally actuated change.

Diver

A diver changes from $I_1 = 10.0 \text{ kg}\cdot\text{m}^2$ at $\omega_1 = 1.5 \text{ rad/s}$ to a tucked configuration with $I_2 = 2.5 \text{ kg}\cdot\text{m}^2$. Then $\omega_2 = 6.0 \text{ rad/s}$. Extending again before entering the water increases I and reduces ω , allowing control of orientation.

Planets, gyroscopes, and reaction wheels

Gravity on a planet is a central force directed along the radius to the Sun, so its torque about the Sun is zero. Angular momentum conservation leads to Kepler's second law: equal areas are swept in equal times, and the planet moves faster near the Sun.

A spinning gyroscope has angular momentum along its spin axis. A torque perpendicular to $\vec{L}$ changes primarily its direction, producing precession rather than simply tipping it over. Reaction wheels exploit internal exchange of angular momentum: accelerating a wheel in one direction rotates the spacecraft body in the opposite direction while total angular momentum remains constant.

13. Coupled translation and rotation: massive-pulley Atwood machine

Two masses m_1 and m_2 are connected by a light inextensible string over a pulley of radius R and moment of inertia I . Unlike an ideal massless pulley, the massive pulley requires torque to accelerate, so the tensions on its two sides are different. Let $m_2 > m_1$, with m_2 moving downward and m_1 upward with acceleration magnitude a .

Equations for the masses

$$T_1 - m_1g = m_1a,$$

$$m_2g - T_2 = m_2a.$$

Torque equation for the pulley

Taking the rotation produced by m_2 as positive,

$$(T_2 - T_1)R = I\alpha.$$

Because the string does not slip, the tangential acceleration of the pulley rim equals the string acceleration:

$$a = \alpha R.$$

Eliminating T_1 , T_2 , and α gives

$$a = (m_2 - m_1)g / (m_1 + m_2 + I/R^2).$$

The extra term I/R^2 acts like an additional effective mass. It reduces acceleration because some gravitational potential energy becomes rotational kinetic energy of the pulley. In the limit $I \rightarrow 0$, the familiar massless-pulley result is recovered.

Numerical example

Let $m_1 = 1.0$ kg, $m_2 = 3.0$ kg, and let the pulley be a solid disk of $M = 2.0$ kg and $R = 0.10$ m. Then $I = \frac{1}{2}MR^2 = 0.010$ kg·m², so $I/R^2 = 1.0$ kg. The acceleration is

$$a = (3.0 - 1.0)(9.81)/(1.0 + 3.0 + 1.0) = 3.92 \text{ m/s}^2.$$

The tensions are $T_1 = m_1(g + a) = 13.73$ N and $T_2 = m_2(g - a) = 17.67$ N. Their difference provides the pulley torque. If the pulley were massless, acceleration would be 4.91 m/s² and the tensions would be equal.

14. Engineering applications and common mistakes

Application	Rotational principle	Design significance
Flywheel	$K_{\text{rot}} = \frac{1}{2}I\omega^2$	Stores energy and smooths power fluctuations.
Turbine	$P = \tau\omega$	Converts fluid energy into shaft power.
Gears	Torque–speed trade-off	Matches motor speed and load requirements.
Robot joint	$\tau = I\alpha$	Controls acceleration and positioning precision.
Satellite	Conservation of L	Reaction wheels provide attitude control.
Electric motor	Magnetic torque	Converts electrical energy into rotation.

Frequent errors

- Treating centrifugal force as a real force in an inertial-frame FBD.
- Drawing “centripetal force” as an extra force instead of the radial resultant of real forces.
- Confusing force with torque or ignoring the lever arm.
- Mixing ω with α and their units rad/s and rad/s^2 .
- Assuming moment of inertia depends only on total mass rather than mass distribution and axis.
- Assuming larger ω always means larger L without considering I .
- Using linear kinematic equations for angular variables without the correct correspondence.
- Ignoring rotational kinetic energy when an object both translates and rotates.
- Applying the parallel-axis or perpendicular-axis theorem to the wrong axes.

Compact problem-solving checklist

- Choose the object or system and specify the rotation axis.
- Draw all real external forces and determine their torque arms.
- Choose positive rotation and apply the right-hand rule consistently.
- Decide whether the problem requires kinematics, $\Sigma\tau = I\alpha$, energy, or angular momentum.
- Include both translational and rotational terms when motion is coupled.
- Check units, signs, limiting cases, and physical plausibility.

15. Summary and review problems

Essential equations

$$s = r\theta, \quad v = r\omega, \quad a_t = r\alpha, \quad a_r = v^2/r = r\omega^2,$$

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}, \quad \Sigma\tau = I\alpha, \quad I = \int r^2 dm,$$

$$K_{\text{rot}} = \frac{1}{2}I\omega^2, \quad P = \tau\omega, \quad \mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \Sigma\tau_{\text{ext}} = dL/dt.$$

Short review problems

- A wheel turns through 12 revolutions in 8.0 s. Find its average angular velocity.
- A 0.30 m radius wheel rotates at 40 rad/s. Find the rim speed and centripetal acceleration.
- A 1200 kg car takes a flat 60 m radius curve with $\mu_s = 0.65$. Find the maximum safe speed.
- A 35 N force acts perpendicular to a 0.75 m wrench. Find the torque.
- A solid disk of mass 4.0 kg and radius 0.20 m experiences 2.4 N·m net torque. Find α .
- A skater reduces her moment of inertia from 5.0 to 2.0 kg·m² while initially rotating at 1.8 rad/s. Find the final angular speed.
- A 10 N·m torque acts through 15 rad. Calculate the work.

Answers for self-checking

- 1) $3\pi \text{ rad/s} \approx 9.42 \text{ rad/s}$. 2) $v = 12 \text{ m/s}$ and $a_r = 480 \text{ m/s}^2$. 3) $v_{\text{max}} = \sqrt{(0.65 \times 9.81 \times 60)} = 19.6 \text{ m/s}$. 4) $26.25 \text{ N}\cdot\text{m}$. 5) $I = \frac{1}{2}MR^2 = 0.080 \text{ kg}\cdot\text{m}^2$ and $\alpha = 30 \text{ rad/s}^2$. 6) $\omega_2 = 4.5 \text{ rad/s}$. 7) 150 J.
- 2)

Final takeaway: Rotation mirrors translation, but geometry and mass distribution matter. Correct axes, real forces, torque arms, and vector directions turn a complicated rotating system into a small set of dependable equations.