

CIRCULAR AND ROTATIONAL MOTION

Kinematics. Dynamics. Energy. Momentum.
Applications.



By the end of this lecture you will be able to:

- Describe circular motion using angular quantities.
- Relate linear and angular motion.
- Apply Newton's laws to rotational motion.
- Understand work, energy, and momentum in rotation.
- Solve real-world engineering problems involving rotation.



Rotational
Kinematics



Rotational
Dynamics



Energy and
Momentum



Engineering
Applications



1. Review: Work, Energy, Momentum and Impulse

★ **Key idea:** Force acting through displacement transfers energy. Force acting over time changes momentum.

Work and energy

1 General work:

$$W = \int_i^f \vec{F} \cdot d\vec{r}$$

For a constant force over straight displacement:

$$W = Fd \cos \theta$$

2 Work–kinetic–energy theorem:

$$W_{\text{net}} = \Delta K = K_f - K_i$$

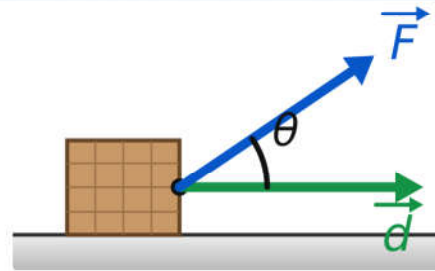
$$K = \frac{1}{2}mv^2$$

3 Mechanical energy:

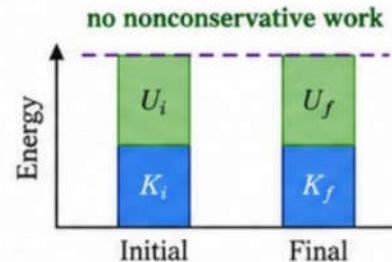
$$E_{\text{mech}} = K + U$$

$$W_{\text{nc}} = \Delta E_{\text{mech}} = \Delta(K + U)$$

If $W_{\text{nc}} = 0$: $K_i + U_i = K_f + U_f$



Same force,
different effect

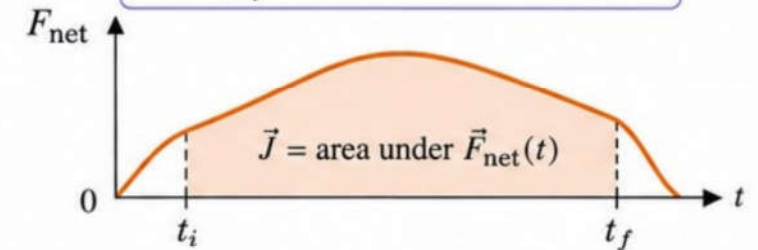


Impulse and linear momentum

1 Linear momentum: $\vec{p} = m\vec{v}$ $[p] = \text{kg} \cdot \text{m} \cdot \text{s}^{-1}$

2 Impulse–momentum theorem:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}_{\text{net}} dt = \Delta \vec{p} = \vec{p}_f - \vec{p}_i$$



3 Momentum conservation:

If $\sum \vec{F}_{\text{ext}} = 0$: $\vec{P}_i = \vec{P}_f$

Total momentum is conserved for an isolated system.

Quantity	Work–energy	Impulse–momentum
Nature	scalar	vector
Accumulation	area under $F_{\text{parallel}}-x$ graph	area under $F_{\text{net}}-t$ graph
Change obtained	ΔK or ΔE_{mech}	$\Delta \vec{p}$
Conservation condition	$W_{\text{nc}} = 0$ for mechanical energy	$\sum \vec{F}_{\text{ext}} = 0$ for total momentum



Take-home message: Rotational dynamics uses the same structure: torque changes angular momentum, and rotational work changes rotational energy.

2. Why Circular and Rotational Motion Matter



Key takeaway: Circular and rotational motion appear throughout nature, technology, and engineering. Understanding them is essential for explaining motion, designing components, and ensuring safety and efficiency.

Circular and rotational motion in real-world situations

1 Transportation



- Cars turning on curved roads
- Trains on curved tracks
- Airplanes banking in turns

→ Safe design requires correct centripetal acceleration and force.

2 Rotating Machinery



- Turbine and compressor rotors
- Electric motors and generators
- Pumps, fans, and mixers

→ Performance, strength, and efficiency depend on rotation.

3 Everyday Life



- Ferris wheels
- Washing-machine spin cycles
- Ceiling fans and bicycle wheels

→ Familiar examples governed by rotational mechanics.

4 Sports



- Figure-skating spins
- Gymnastics and diving rotations
- Discus, hammer, and ball sports

→ Torque and angular momentum control performance.

5 Engineering Systems



- Gyroscopes and IMUs
- Robotics and drones
- Satellite attitude control

→ Rotation provides control, stability, and precision.

6 Nature and Space



- Planets follow nearly elliptical orbits
 - Moons orbit planets
 - Stars and galaxies rotate
- Circular motion is an important approximation; real orbits need not be perfect circles.



Why study it?

- ✓ Many systems involve rotation; ignoring it can lead to incorrect designs.
- ✓ It connects linear motion with angular motion.
- ✓ It introduces torque, moment of inertia, and angular momentum.
- ✓ It helps analyze energy, stability, and performance.



What we will learn in this lecture

- Describe motion using angular quantities.
- Relate linear and angular quantities.
- Apply torque and Newton's second law for rotation.
- Use rotational work-energy and angular momentum.
- Solve engineering problems involving rotation.



Big picture

From tiny rotors to massive turbines and orbiting planets, rotational mechanics unifies many problems in engineering and nature.



Take-home message: Mastering circular and rotational motion helps us understand, predict, and design systems that turn, spin, and orbit safely and efficiently.



Next: Angular Position and Angular Displacement

3. Angular Position and Angular Displacement

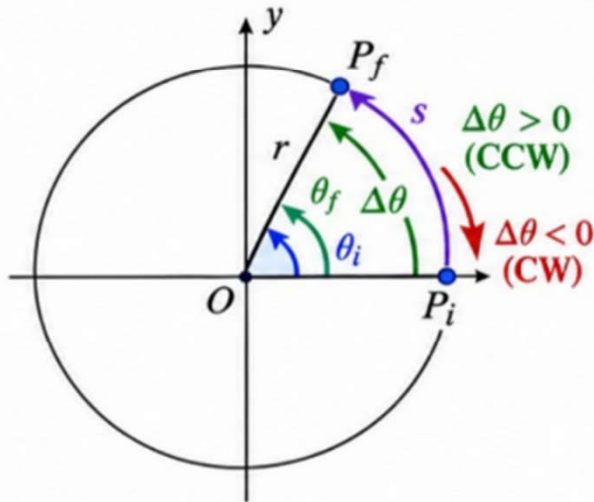


Key idea

Angular position specifies where a point lies on a circle. Angular displacement measures the signed change in angular position.

$$\Delta\theta = \theta_f - \theta_i$$

1. Geometry and sign convention



- Positive rotation: **counterclockwise**
- Negative rotation: **clockwise**
- Angles are measured from a chosen reference direction, usually $+x$.

2. Angular position

$$\theta = \frac{s}{r}$$

$$[\theta] = \text{rad}$$

An angle of 1 rad subtends an arc equal in length to the radius.

$$2\pi \text{ rad} = 360^\circ = 1 \text{ revolution}$$

θ (rad)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
θ ($^\circ$)	0°	90°	180°	270°	360°

Use radians in rotational-motion equations.

3. Angular displacement and arc length

$$\Delta\theta = \theta_f - \theta_i$$

$$s = r\Delta\theta$$

$$\Delta\theta = \frac{s}{r}$$

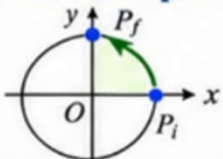
The relation $s = r\Delta\theta$ requires $\Delta\theta$ in radians.

- Angular position may be represented modulo 2π .
- Angular displacement records the complete signed rotation and may exceed 2π .

Examples

A

Positive displacement

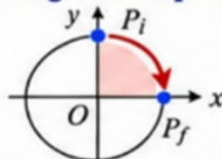


$$\theta_i = 0^\circ, \theta_f = 90^\circ$$

$$\Delta\theta = +90^\circ = +\frac{\pi}{2} \text{ rad}$$

B

Negative displacement

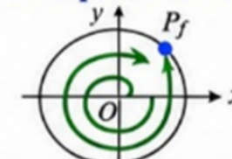


$$\theta_i = 90^\circ, \theta_f = 0^\circ$$

$$\Delta\theta = -90^\circ = -\frac{\pi}{2} \text{ rad}$$

C

Multiple revolutions



$$\theta_i = 0^\circ$$

$$\text{Total rotation} = 810^\circ \text{ (CCW)}$$

$$\Delta\theta = 810^\circ = \frac{9\pi}{2} \text{ rad} = 2.25 \text{ revolutions}$$



Take-home message: Angular position tells where the point is. Angular displacement tells how far and in which rotational direction it moved.

4. Angular Velocity



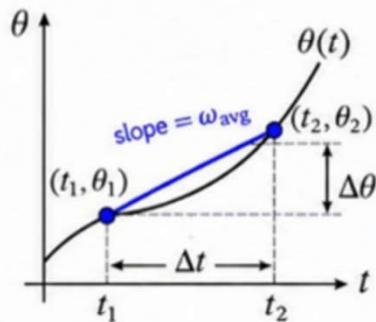
Key idea

Angular velocity describes how rapidly angular position changes and specifies the axis and sense of rotation.

$$\omega = \frac{d\theta}{dt}$$

1. Average and instantaneous angular velocity

Average over an interval

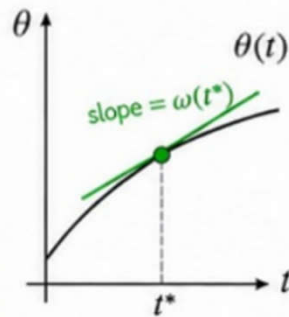


$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

$$\omega(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

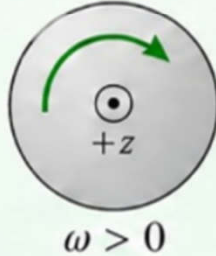
Average angular velocity applies to an interval.
Instantaneous angular velocity applies at one time.

Instantaneous at time t^*



2. Direction and sign

A. Counterclockwise rotation viewed from +z



B. Clockwise rotation viewed from +z



Right-hand rule

Curl the fingers of your right hand in the sense of rotation. Your thumb gives the direction of $\vec{\omega}$.



Fingers curl counterclockwise, thumb points along +z.

$$\vec{\omega} = \omega \hat{k}$$

The sign of the scalar ω depends on the chosen positive axis.

3. Uniform angular velocity

For rotation about a fixed axis with constant angular velocity:

$$\omega = \text{constant}$$

$$\theta(t) = \theta_0 + \omega t$$

$$\Delta\theta = \omega \Delta t$$

Period and frequency

$$\omega = \frac{2\pi}{T} = 2\pi f$$

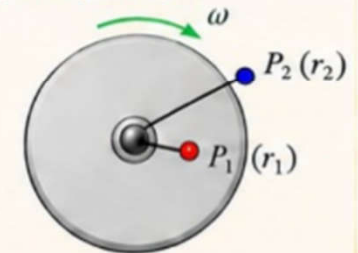
$$f = \frac{1}{T}$$

$$[\omega] = \text{rad s}^{-1}$$

$$[f] = \text{Hz}$$

$$[T] = \text{s}$$

Rigid body example



Every point of a rigid body has the same angular velocity, although tangential speed depends on radius.



Interpretation

- Angular velocity is an axial vector directed along the rotation axis.
- Its magnitude gives the rate of change of angular position.
- For fixed-axis rotation, positive and negative signs specify the sense of rotation.
- One complete revolution corresponds to $2\pi \text{ rad}$.



Worked check

A wheel rotates from $\theta_1 = 45^\circ$ to $\theta_2 = 200^\circ$ in 2.0 s. Find its average angular velocity.

$$\omega_{\text{avg}} = \frac{200^\circ - 45^\circ}{2.0 \text{ s}} = 77.5^\circ \text{ s}^{-1} = 1.35 \text{ rad s}^{-1}$$

positive, counterclockwise



Take-home message:

Angular velocity gives both the rate of rotation and the direction of the rotation axis.

5. Angular Acceleration



Key takeaway: Angular acceleration is the rate of change of angular velocity. For rotation about a fixed axis, its sign shows whether ω increases or decreases relative to the chosen positive direction.

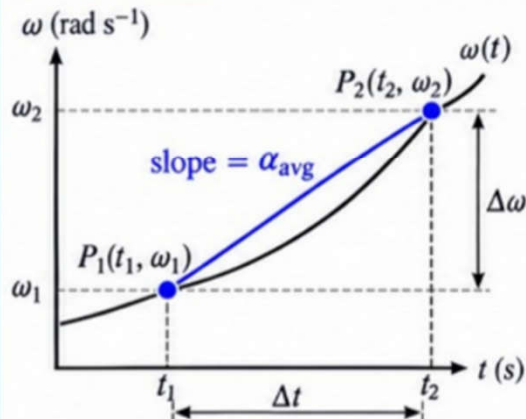
$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

$[\alpha] = \text{rad s}^{-2}$

1. Average angular acceleration

The average angular acceleration over the interval $\Delta t = t_2 - t_1$ is

$$\alpha_{\text{avg}} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

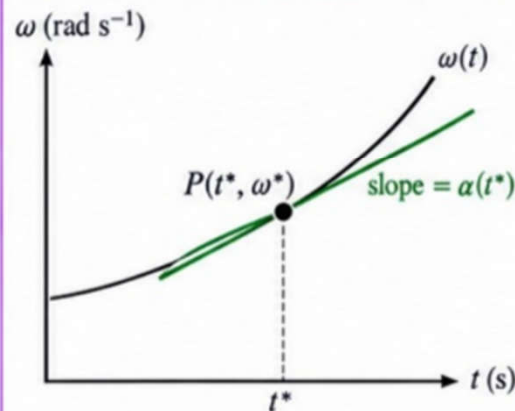


Average angular acceleration belongs to a finite time interval.

2. Instantaneous angular acceleration

At time t^* , the instantaneous value is the slope of the tangent to $\omega(t)$.

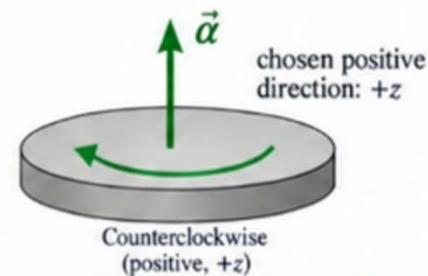
$$\alpha(t^*) = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} = \left. \frac{d\omega}{dt} \right|_{t^*}$$



Instantaneous angular acceleration belongs to one instant.

3. Direction and sign

For fixed-axis rotation, $\vec{\alpha}$ points along the rotation axis. The right-hand rule defines the positive direction.



Speed increases when ω and α have the same sign.

ω	α	Result
$\omega > 0$	$\alpha > 0$	Speeds up counterclockwise
$\omega > 0$	$\alpha < 0$	Slows down counterclockwise
$\omega < 0$	$\alpha < 0$	Speeds up clockwise
$\omega < 0$	$\alpha > 0$	Slows down clockwise

4. Constant angular acceleration

If α is constant and rotation is about a fixed axis:

$$\omega = \omega_0 + \alpha t$$

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2} \alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

$$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$$

Linear-rotational analogy

$$x \leftrightarrow \theta \quad v \leftrightarrow \omega \quad a \leftrightarrow \alpha$$

Symbols:

θ angular position (rad)
 ω angular velocity (rad s⁻¹)
 α angular acceleration (rad s⁻²)
 Subscript 0: value at $t = 0$



Key points:

- Average α uses a secant slope over an interval.
- Instantaneous α uses the tangent slope at one time.
- Positive and negative α refer to the chosen axis direction.



Physical interpretation

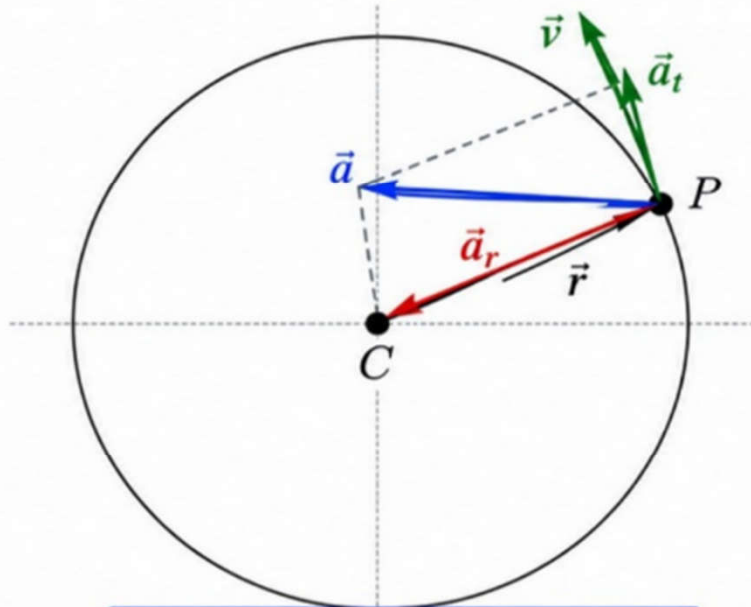
Angular acceleration determines how quickly rotational speed changes. When α is opposite to ω , the rotation slows; when α has the same sign as ω , it speeds up.

★ **Take home:** Angular acceleration describes how angular velocity changes in magnitude or direction.

6. Circular Motion: Acceleration Vectors



Tangential acceleration changes speed; radial acceleration changes direction.



Shown case: CCW and speeding up
 $\vec{\omega}$ and $\vec{\alpha}$ point along $+z$



Uniform circular motion

$$\alpha = 0 \Rightarrow a_t = 0$$

$$a_r = \frac{v^2}{r} \neq 0$$

Speed is constant;
 direction changes.

Instantaneously at rest

$$\omega = 0 \Rightarrow a_r = 0$$

$$\alpha \text{ may be nonzero} \Rightarrow$$

$$a_t \text{ may be nonzero}$$

Zero speed at one instant
 does not require zero
 acceleration.

1 Tangential component

$$\vec{a}_t = \vec{\alpha} \times \vec{r}$$

Magnitude for fixed-axis rotation:

$$a_t = r |\alpha|$$

Direction:

Tangent to the path. Its sense follows the sign of α relative to the chosen axis.

$\vec{a}_t$ is parallel to $\vec{v}$ when speed increases and antiparallel when speed decreases.

2 Radial (centripetal) component

Magnitude:

$$a_r = \frac{v^2}{r} = \omega^2 r$$

Vector equation:

$$\vec{a}_r = -\omega^2 \vec{r}$$

Direction:

Always toward the center for $r > 0$.

The scalar a_r is a nonnegative magnitude; the minus sign in the vector equation gives the inward direction.

3 Total acceleration

$$\vec{a} = \vec{a}_t + \vec{a}_r$$

Because $\vec{a}_t \perp \vec{a}_r$:

$$|\vec{a}| = \sqrt{a_t^2 + a_r^2}$$

Direction:

The total vector lies between the inward radial direction and the tangential direction.

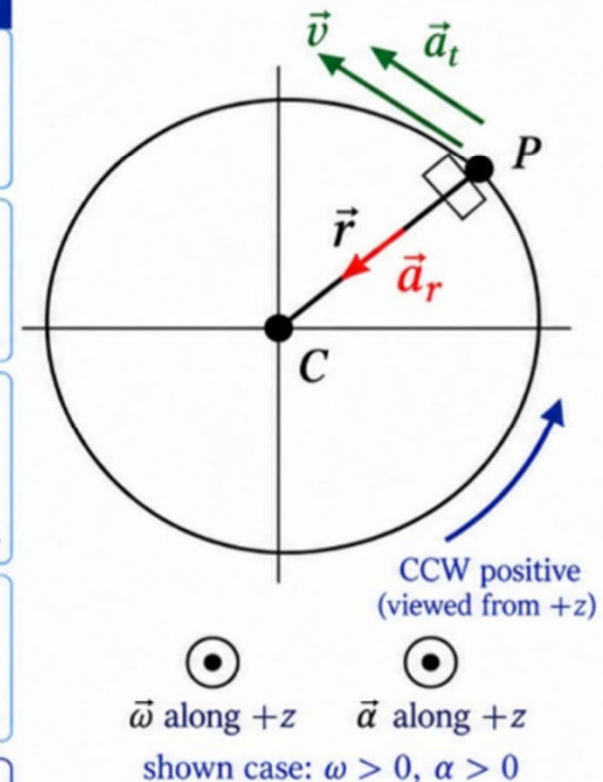


Constant speed does not mean zero acceleration: in uniform circular motion, $\vec{a} = \vec{a}_r$ points inward.

7. Linear and Angular Motion: Core Analogies

★ Angular quantities describe rotation about an axis; linear quantities describe the motion of a point at radius r .

	LINEAR: point motion	ANGULAR: rotation about an axis	RADIUS RELATION: fixed r
1 POSITION	s (signed arc coordinate)	θ (angular position in radians)	$s = r\theta$ θ must be in radians.
2 VELOCITY	$v_t = \frac{ds}{dt}$ (signed tangential component)	$\omega = \frac{d\theta}{dt}$ (signed angular velocity)	$v_t = r\omega$ $v = v_t = r \omega $ v is speed; $\vec{v}$ is tangent to the path.
3 TANGENTIAL ACCELERATION	$a_t = \frac{dv_t}{dt}$ (signed tangential component)	$\alpha = \frac{d\omega}{dt}$ (signed angular acceleration)	$a_t = r\alpha$ $ a_t = r \alpha $ The sign sets the tangential sense.
4 RADIAL ACCELERATION	$a_r = \frac{v^2}{r}$ (centripetal magnitude)	ω^2 (direction independent of the sign of ω)	$a_r = \omega^2 r$ $\vec{a}_r = -\omega^2 \vec{r}$ Always directed toward the center.
★ Key distinctions	s, v_t, a_t, and a_r describe the motion of a point on the body. θ, ω, and α describe rotation about the axis. - For fixed r, larger r gives larger v and a_t at the same ω and α. - Radial acceleration depends on ω^2, so it points inward for either sense of rotation.		



Take-home message: One rotational motion produces different linear speeds and accelerations at different radii.

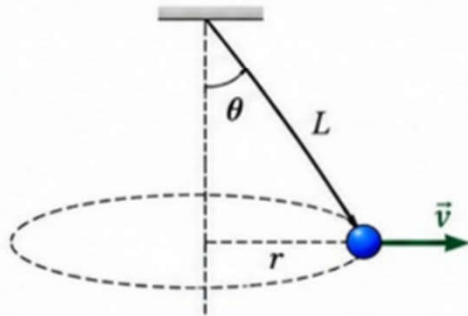
8. Examples of Centripetal Force

The centripetal force is not an additional force. It is the net inward component of the real forces acting on the object.

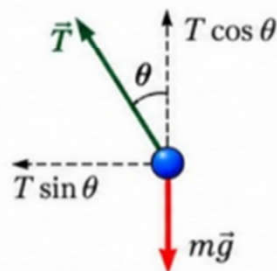
$$\sum F_r = ma_r = \frac{mv^2}{r}$$

(a) Conical pendulum

Physical situation



Free-body diagram (isolated mass)



Equations

Vertical: $T \cos \theta - mg = 0$

Radial inward: $T \sin \theta = \frac{mv^2}{r}$

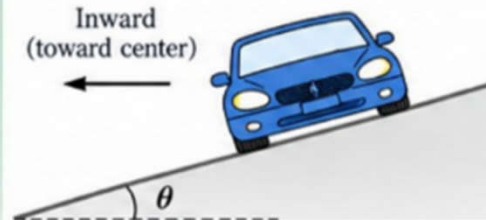
Therefore: $\tan \theta = \frac{v^2}{rg}$

Interpretation

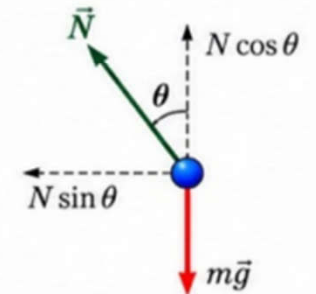
The horizontal component of tension supplies the inward net force.

(b) Frictionless banked curve

Physical situation



Free-body diagram (isolated car)



Equations

Vertical: $N \cos \theta - mg = 0$

Radial inward: $N \sin \theta = \frac{mv^2}{r}$

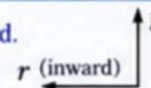
Therefore: $\tan \theta = \frac{v^2}{rg}$

Design speed: $v = \sqrt{rg \tan \theta}$

Interpretation

The horizontal component of the normal force supplies the inward net force.

Choose +r inward and +y upward.



Key idea: Different real forces can produce the same radial acceleration.
Always draw the real forces first, then project their components along the inward radial direction.

r : radius of path

v : speed

θ : string or bank angle

T : tension

N : normal force

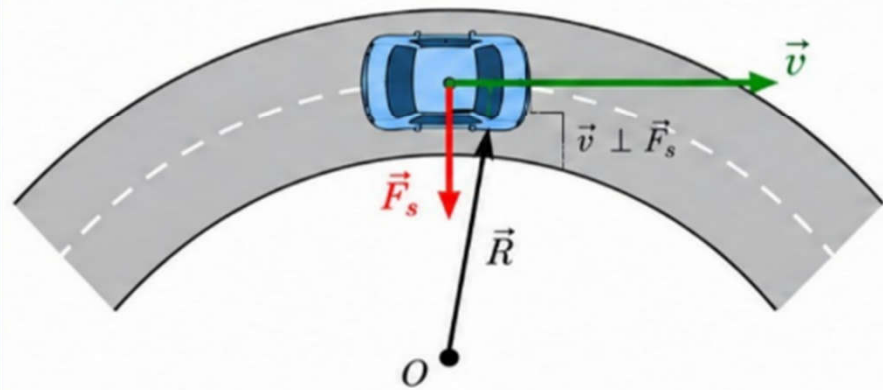
m : mass

g : acceleration due to gravity

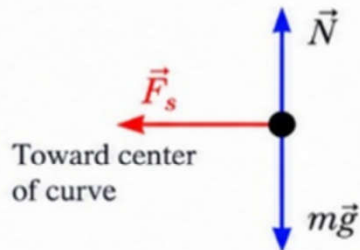
9. Flat Curve with Friction

A car of mass m travels at speed v around a level curve of radius R . Static friction between the tires and the road supplies the required inward force.

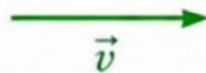
Top view of the curved path



Free-body diagram of the car



Velocity is tangent to the path; it is not a force.



Equations and physical meaning

1. Vertical balance (no vertical acceleration)

$$\Sigma F_y = N - mg = 0 \Rightarrow N = mg$$

2. Radial dynamics (inward toward the center)

$$\Sigma F_r = F_s = \frac{mv^2}{R}$$

F_s is a real contact force exerted by the road on the tires.

“Centripetal force” names the net inward force; it is not an additional force.

3. Static-friction condition

$$0 \leq F_s \leq \mu_s N \quad \text{Therefore,} \quad \frac{mv^2}{R} \leq \mu_s mg$$

$$\text{No-slip condition:} \quad v^2 \leq \mu_s gR$$

4. Maximum speed before slipping

$$v_{\max} = \sqrt{\mu_s gR}$$

5. Interpretation

- If $v < v_{\max}$, static friction adjusts to $F_s = mv^2/R$ and remains below its maximum.
- At $v = v_{\max}$, $F_s = \mu_s N$ and the car is at the limiting condition.
- If $v > v_{\max}$, the required inward force exceeds the available static friction; the car begins to slide outward relative to the curve.

Symbols

N	: normal force (upward)
mg	: weight (downward)
F_s	: static friction (inward)
m	: car mass
v	: speed of the car
R	: radius of the curve
μ_s	: coefficient of static friction (road–tire)
g	: gravitational acceleration



Key Point

On a level curve, static friction supplies the horizontal inward force. The limiting speed depends on μ_s , g , and R , but not on the car's mass.

10. Torque: Definition and Magnitude



Key takeaway: Torque measures the rotational effect of a force about a chosen point or axis.

Definition:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$\vec{r}$ points from the axis O to the force-application point P .

Magnitude:

$$\tau = rF \sin \phi$$

ϕ is the angle between $\vec{r}$ and $\vec{F}$, with $0^\circ \leq \phi \leq 180^\circ$.

Lever-arm form:

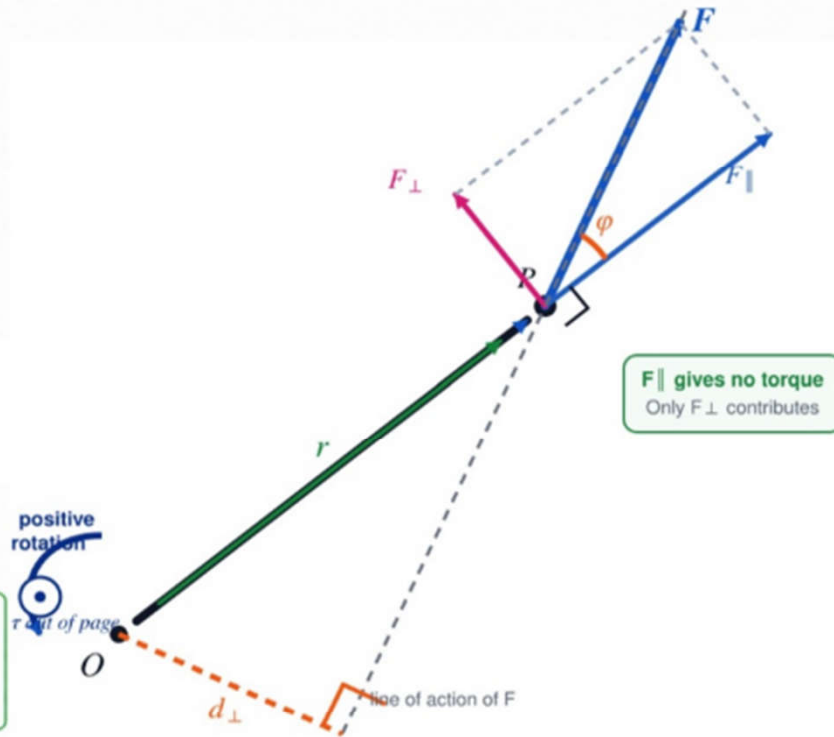
$$d_\perp = r \sin \phi$$

$$\tau = F d_\perp$$



Only the component of force perpendicular to $\vec{r}$ produces torque:

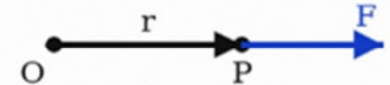
$$F_\perp = F \sin \phi$$



Special cases

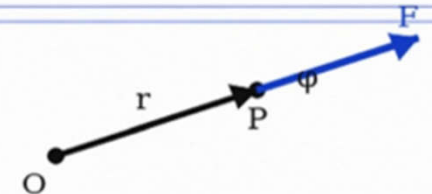
1 $\phi = 0^\circ$ or 180°

F parallel/antiparallel to $\vec{r}$;
line of action passes through O ; $\tau = 0$.



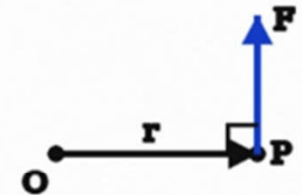
2 $0^\circ < \phi < 90^\circ$

Partial torque;
 $0 < \tau < rF$.



3 $\phi = 90^\circ$

F perpendicular to $\vec{r}$;
 $\tau = rF$,
maximum magnitude for fixed r and F .



Vector definition:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Magnitude:

$$\tau = rF \sin \phi = F d_\perp$$

Unit:

$$\text{N} \cdot \text{m}$$

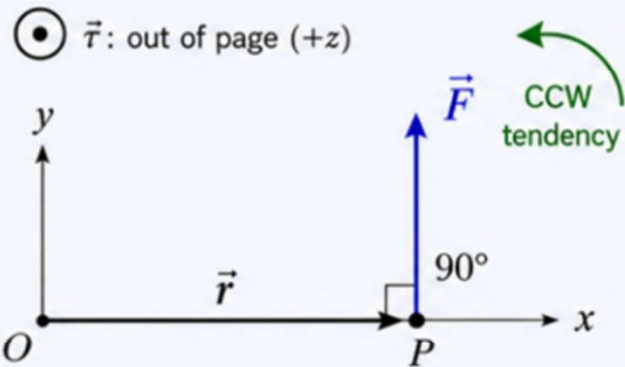
Clarification:

Torque has the same dimensions as energy, but torque is not measured in joules.

11. Direction of Torque: The Right-Hand Rule

★ **Key takeaway:** The torque vector is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$. Its direction follows the right-hand rule for $\vec{\tau} = \vec{r} \times \vec{F}$.

1. Vector example (viewed from +z)



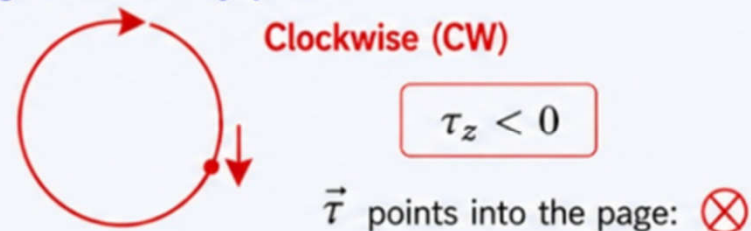
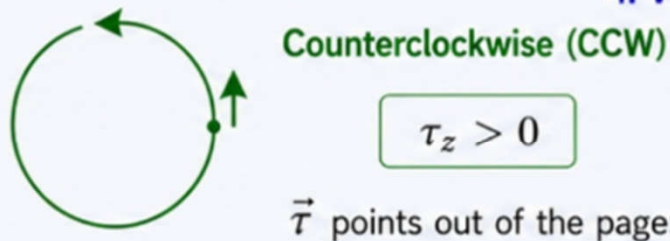
2. Right-hand rule

- 1 Point the fingers of your right hand along $\vec{r}$.
 - 2 Curl them toward $\vec{F}$ through the smaller angle.
 - 3 Your thumb gives the direction of $\vec{\tau}$.
-

3. Notes

- Torque is not a force. It describes the rotational effect of a force about the chosen point or axis.
- Changing the reference point can change the torque.
- Unit: $\text{N}\cdot\text{m}$
- The sign convention is meaningful only after the positive axis/viewing direction is specified.

4. Viewed from +z, looking toward the xy-plane



$$\vec{\tau} = \vec{r} \times \vec{F}$$

CCW viewed from +z $\rightarrow +\tau_z \rightarrow \odot$

CW viewed from +z $\rightarrow -\tau_z \rightarrow \otimes$

2. Why Circular and Rotational Motion Matter?



Key takeaway: Circular and rotational motion appear throughout nature, technology, and engineering. Understanding them is essential for explaining motion, designing components, and ensuring safety and efficiency.

Circular and rotational motion in real-world situations

1 Transportation



- Cars turning on curved roads
- Trains on curved tracks
- Airplanes banking in turns

→ Safe design requires correct centripetal acceleration and force.

2 Rotating Machinery



- Turbine and compressor rotors
- Electric motors and generators
- Pumps, fans, and mixers

→ Performance, strength, and efficiency depend on rotation.

3 Everyday Life



- Ferris wheels
- Washing-machine spin cycles
- Ceiling fans and bicycle wheels

→ Familiar examples governed by rotational mechanics.

4 Sports



- Figure-skating spins
- Gymnastics and diving rotations
- Discus, hammer, and ball sports

→ Torque and angular momentum control performance.

5 Engineering Systems



- Gyroscopes and IMUs
- Robotics and drones
- Satellite attitude control

→ Rotation provides control, stability, and precision.

6 Nature and Space



- Planets follow nearly elliptical orbits
- Moons orbit planets
- Stars and galaxies rotate

→ Circular motion is an important approximation; real orbits need not be perfect circles.



Why study it?

- ✓ Many systems involve rotation; ignoring it can lead to incorrect designs.
- ✓ It connects linear motion with angular motion.
- ✓ It introduces torque, moment of inertia, and angular momentum.
- ✓ It helps analyze energy, stability, and performance.



What we will learn in this lecture

- Describe motion using angular quantities.
- Relate linear and angular quantities.
- Apply torque and Newton's second law for rotation.
- Use rotational work-energy and angular momentum.
- Solve engineering problems involving rotation.



Big picture

From tiny rotors to massive turbines and orbiting planets, rotational mechanics unifies many problems in engineering and nature.



Take-home message: Mastering circular and rotational motion helps us understand, predict, and design systems that turn, spin, and orbit safely and efficiently.



Next: Angular Position and Angular Displacement

13. Moment of Inertia: Definition and Physical Meaning



Key takeaway: Moment of inertia measures how mass is distributed relative to a specified axis of rotation. Larger I means greater resistance to angular acceleration.

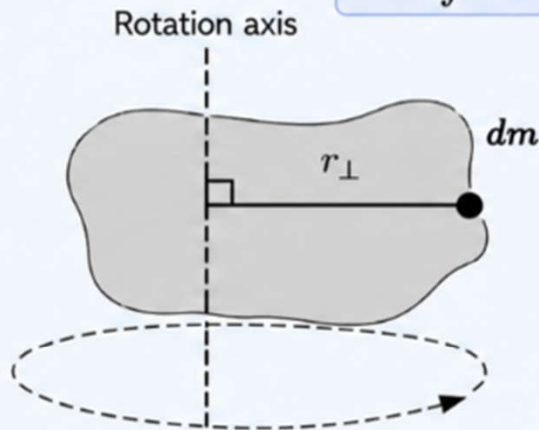
1.1 Definitions

For point particles:

$$I = \sum_i m_i r_i^2$$

For a continuous body:

$$I = \int r_{\perp}^2 dm$$



r_i (or $r_{\perp}$) is the shortest perpendicular distance from the mass element to the rotation axis.

Unit: $\text{kg} \cdot \text{m}^2$

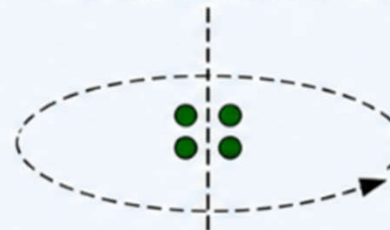
1.2 Physical meaning

Fixed axis, constant I_z :

$$\sum \tau_z = I_z \alpha \quad \rightarrow \quad \alpha = \frac{\sum \tau_z}{I_z}$$

- For the same net torque, larger I gives smaller angular acceleration.
- I depends on the chosen axis and the entire mass distribution, not only total mass.
- Contributions grow with the square of distance: doubling r makes a mass element contribute four times as much.

A. Mass close to the axis



smaller I

B. Mass farther from the axis



larger I

Same mass, different distribution.



Bottom takeaway: Moment of inertia is the rotational analogue of mass, but it also depends on where the mass is located relative to the chosen axis.

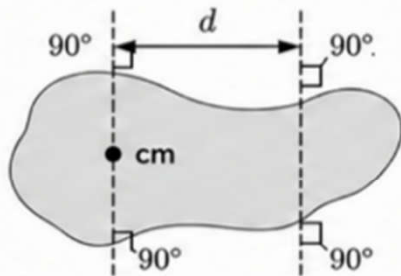
14. Theorems and Common Moments of Inertia

Key takeaway: Moment of inertia depends on both the mass distribution and the chosen axis. Two axis theorems let us relate moments about different axes.

2.1 Parallel-Axis Theorem

Moment of inertia about an axis parallel to an axis through the center of mass:

$$I = I_{cm} + Md^2$$



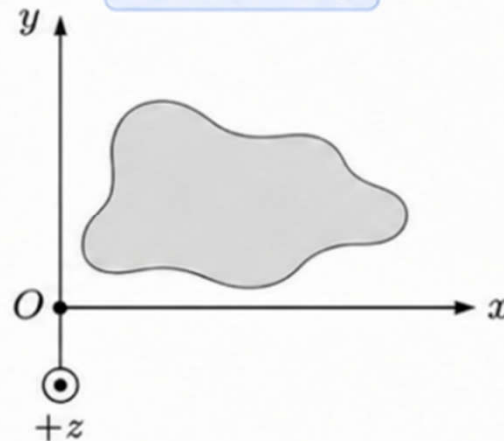
I_{cm} : moment of inertia about an axis through the center of mass
 M : total mass
 d : perpendicular distance between the two parallel axes

✓ Valid for any axis parallel to the center-of-mass axis.

2.2 Perpendicular-Axis Theorem

For a thin planar lamina in the xy -plane:

$$I_z = I_x + I_y$$



✓ Applies only to a planar lamina and mutually perpendicular axes through the same point O .

2.3 Common Moments of Inertia

Object	Axis	Moment of inertia I
1. Point mass at perpendicular distance r	Chosen axis	$I = mr^2$
2. Thin ring, radius R	Through center, perpendicular to plane	$I = MR^2$
3. Solid disk, radius R	Through center, perpendicular to plane	$I = \frac{1}{2} MR^2$
4. Thin rod, length L	Through center, perpendicular to rod	$I = \frac{1}{12} ML^2$
5. Thin rod, length L	Through one end, perpendicular to rod	$I = \frac{1}{3} ML^2$



Remember: Always identify the rotation axis before selecting a formula.

Unit: $\text{kg} \cdot \text{m}^2$

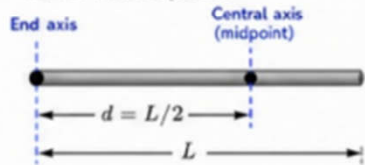
15. Using the Theorems and Real-World Applications

★ Key takeaway: The axis theorems and standard formulas let us calculate moment of inertia and predict angular acceleration.

3.1 Examples – Parallel Axis Theorem

A Thin rod about one end

- Thin uniform rod, length L , mass M .
- Central axis perpendicular to the rod at its midpoint.
- End axis perpendicular to the rod at one end.
- The two axes are parallel.



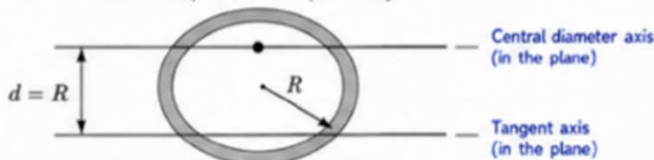
$$I_{cm} = \frac{1}{12} ML^2$$

$$I_{end} = I_{cm} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{1}{3} ML^2$$

B Thin ring about a tangent axis IN ITS PLANE

- Thin ring of radius R , mass M , lying in its plane.
- Central axis is a diameter in the plane of the ring.
- Tangent axis is a line in the plane, tangent to the ring.
- The two axes are parallel and separated by $d = R$.



$$I_{cm} = \frac{1}{2} MR^2$$

$$I_{tangent} = I_{cm} + MR^2$$

$$I_{tangent} = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2$$

3.2 Example – Rotating Solid Disk

A solid disk has $M = 2.0 \text{ kg}$, $R = 0.30 \text{ m}$, and an applied torque $\tau = 1.2 \text{ N} \cdot \text{m}$ about its central axis.

Moment of inertia about the central axis:

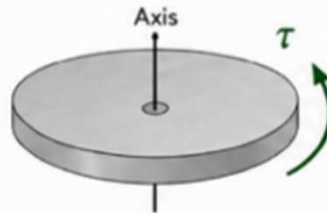
$$I = \frac{1}{2} MR^2$$

$$= \frac{1}{2} (2.0)(0.30)^2$$

$$= 0.090 \text{ kg} \cdot \text{m}^2$$

Angular acceleration:

$$\alpha = \frac{\tau}{I} = \frac{1.2}{0.090} = 13.3 \text{ rad/s}^2$$



Assuming the applied torque is the net torque,
 $\alpha = 13.3 \text{ rad/s}^2$ CCW.

3.3 Real-World Applications



Flywheel

Larger I can store more rotational kinetic energy at the same ω :

$$K_{rot} = \frac{1}{2} I \omega^2.$$



Figure skater

With negligible external torque, pulling arms inward decreases I and increases ω while L remains constant.



Vehicle wheel/tire

Mass farther from the axis increases I , requiring more torque for the same angular acceleration.



Reaction wheel (spacecraft)

A spacecraft reorients by exchanging angular momentum with an internal reaction wheel; it is not primarily changing the spacecraft's moment of inertia.

Key points

- I depends on mass distribution and axis.
- Parallel-axis theorem: $I = I_{cm} + Md^2$.
- Units: $\text{kg} \cdot \text{m}^2$.

Takeaway

Larger torque or smaller I produces larger angular acceleration for a fixed axis.

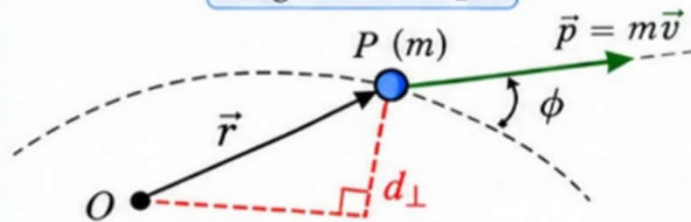
16. Angular Momentum: Definition and Physical Meaning

★ Key takeaway: Angular momentum describes rotational motion about a specified point or axis. It depends on the motion and on the chosen origin or axis.

1.1 Point Particle

A particle of mass m is at position $\vec{r}$ from origin O and moving with velocity $\vec{v}$ (momentum $\vec{p} = m\vec{v}$).

$$\vec{L}_O = \vec{r} \times \vec{p}$$



$$|\vec{L}_O| = rp \sin \phi = mrv \sin \phi = pd_{\perp}$$

where $d_{\perp} = r \sin \phi$.

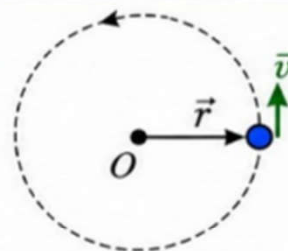
$$\text{If } \vec{p} \perp \vec{r} (\phi = 90^\circ): |\vec{L}_O| = mrv.$$

Changing the origin O generally changes $\vec{r}$ and therefore $\vec{L}_O$.

1.2 Direction of $\vec{L}_O$ (Right-Hand Rule)

Curl the fingers from $\vec{r}$ toward $\vec{p}$. The thumb gives the direction of $\vec{L}_O$.

Counterclockwise (CCW)

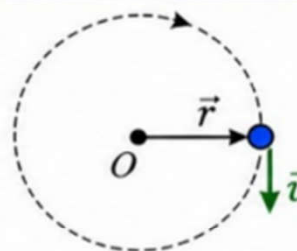


$$\vec{L}_O = +L_z$$

(out of the page)
+z direction

$\vec{L}_O$ points out of the page
(+z direction).

Clockwise (CW)



$$\vec{L}_O = -L_z$$

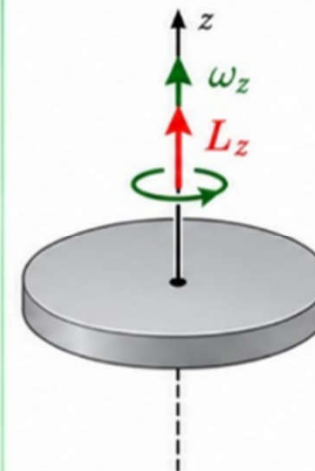
(into the page)
-z direction

$\vec{L}_O$ points into the page
(-z direction).

1.3 Rigid Body about a Fixed Principal Axis

For rotation about a fixed principal (or symmetry) axis z :

$$L_z = I_z \omega_z$$



I_z : moment of inertia
about the z-axis

ω_z : signed angular
velocity

L_z : angular-momentum
component along z

For rotation about a fixed principal
axis, $\vec{L}$ is parallel to $\vec{\omega}$.

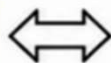
More generally, $\vec{L} = \mathbf{I} \cdot \vec{\omega}$ and
 $\vec{L}$ need not be parallel to $\vec{\omega}$.

Translational and Rotational Analogy

Linear (translation)

$$\vec{p} = m\vec{v}$$

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}$$



Rotational (about a point O)

$$\vec{L}_O = \vec{r} \times \vec{p}$$

$$\vec{\tau}_{O,\text{ext}} = \frac{d\vec{L}_O}{dt}$$

For a fixed principal axis (constant I_z):

$$\tau_z = I_z \alpha_z$$

Units

$$[L] = \text{kg m}^2 \text{s}^{-1}$$

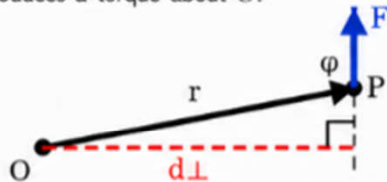
Physical Meaning

- Larger mass, speed, or perpendicular distance from the axis gives larger angular momentum.
- Angular momentum depends on the selected origin or rotation axis.
- Net external torque changes angular momentum.

17. Torque Changes Angular Momentum

2.1 From Force to Torque

A force $\vec{F}$ applied at point P relative to point O produces a torque about O .



$$\vec{\tau}_O = \vec{r} \times \vec{F}$$

$$\tau_O = rF \sin \phi = Fd_{\perp}$$

- ϕ is the angle between $\vec{r}$ and $\vec{F}$.
- $d_{\perp}$ is the shortest perpendicular distance from O to the line of action of $\vec{F}$.

2.2 Fundamental Law

The net external torque about O equals the time rate of change of angular momentum about O .

$$\sum \vec{\tau}_{O,\text{ext}} = \frac{d\vec{L}_O}{dt}$$

This is the general rotational form of Newton's second law.

Fixed principal axis, constant I_z :

$$L_z = I_z \omega_z$$

$$\sum \tau_{\text{ext},z} = I_z \alpha_z$$

If I changes:

Use $\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$; do not replace it by $I\alpha$ without checking the assumptions.

2.3 Translational / Rotational Analogy

Translational (Linear)

Kinematic quantity

$$\vec{p} = m\vec{v}$$

Dynamics (general)

$$\sum \vec{F}_{\text{ext}} = \frac{d\vec{p}}{dt}$$

Fixed mass m

$$\sum \vec{F}_{\text{ext}} = m\vec{a}$$

Rotational (About Point O)

Kinematic quantity

$$\vec{L}_O = \vec{r} \times \vec{p}$$

Dynamics (general)

$$\sum \vec{\tau}_{O,\text{ext}} = \frac{d\vec{L}_O}{dt}$$

Fixed principal axis, constant I

$$\sum \tau_{z,\text{ext}} = I_z \alpha_z$$

2.4 What Torque Does (viewed from +z toward xy-plane)

Positive Net Torque
(Counterclockwise)



$$\sum \tau_z > 0$$

$\rightarrow L_z$ increases.

Negative Net Torque
(Clockwise)



$$\sum \tau_z < 0$$

A negative net torque makes L_z more negative. It reduces $|L_z|$ only when $L_z > 0$.

2.5 Examples



- Pushing a door farther from the hinge increases torque.



- Braking applies a torque opposite to angular momentum.



- Steering or gyroscopic torque can change the *direction* of angular momentum.

2.6 Key Points

- ✓ $\sum \vec{\tau}_{O,\text{ext}} = \frac{d\vec{L}_O}{dt}$ is the most general law.
- ✓ If $\sum \vec{\tau}_{O,\text{ext}} = \vec{0}$, then $\vec{L}_O$ is constant.
- ✓ When I is constant about a fixed principal axis, $\sum \tau_{z,\text{ext}} = I_z \alpha_z$.
- ✓ Units of torque: N·m
- ✓ Units of angular momentum: kg·m²/s

2.7 Notation & Conventions

- Subscript O indicates the chosen reference point.
- "ext" denotes external torques/forces.
- Right-hand rule for $\vec{r} \times \vec{F}$.
- Positive z is out of the page; we view from +z toward the xy -plane.



Take-home message: Torque is what changes angular momentum.
Zero net external torque means angular momentum is conserved.



Next: Conservation of Angular Momentum

18. Conservation of Angular Momentum



Key takeaway: If the net external torque on a system is zero, the total angular momentum vector remains constant.

1) Fundamental vector law

The fundamental vector law:

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$



$$\sum \vec{\tau}_{ext} = 0 \Rightarrow \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{constant}$$

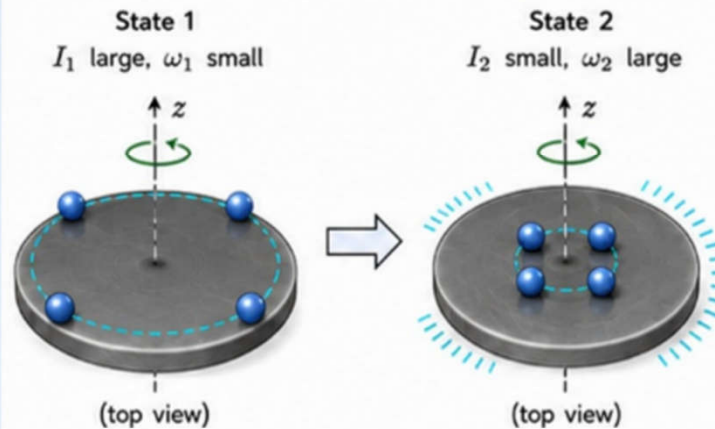
In an inertial frame, both **magnitude** and **direction** remain constant.

2) Fixed-axis special case (about a fixed axis)

For rotation about a fixed axis (constant direction of $\vec{L}$):

$$L_z = I_z \omega_z$$

$$I_1 \omega_1 = I_2 \omega_2$$



If L_z is conserved: **smaller** $I \rightarrow$ **larger** $|\omega|$.

3) Conditions and limits

When is it conserved?

- Net external torque about the chosen origin is zero
- External torques cancel or are negligible over the interval
- Internal forces may redistribute angular momentum, but cannot change the system total

It is **NOT** conserved (in general):

$$\text{If } \sum \vec{\tau}_{ext} \neq 0,$$

magnitude and/or direction of $\vec{L}$ changes.



Common mistake:

Angular velocity is not generally conserved. It stays constant only if both L and I stay constant.



Take-home message: When the net external torque on a system is zero, the total angular momentum is conserved. This principle explains many important phenomena in nature and technology.



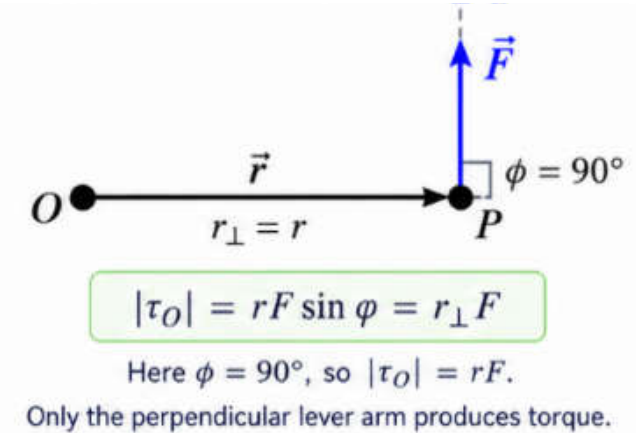
Next: Conservation in Action – Figure Skater and Diver

19. Translational and Rotational Dynamics

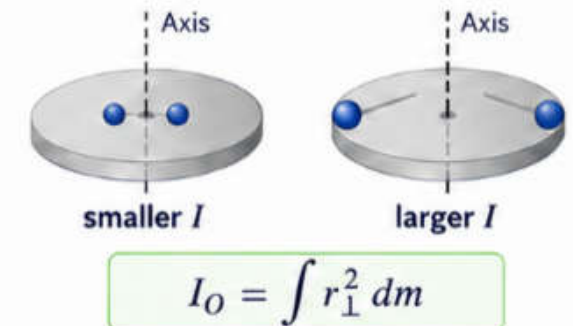
Core analogies from mass and force to momentum, energy, work, and power.

Dynamical and Energetic Analogies

	TRANSLATION	PHYSICAL ROLE	ROTATION ABOUT AN AXIS
Inertia	Mass m kg	Resistance to a change in motion	Moment of inertia $I_O = \int r_{\perp}^2 dm$ $\text{kg} \cdot \text{m}^2$
Cause of motion change	Force $\vec{F}$	Changes linear momentum	Torque $\vec{\tau}_O = \vec{r} \times \vec{F}$ $ \tau_O = r_{\perp} F$
Momentum	$\vec{p} = m\vec{v}$	State of translational motion	$\vec{L}_O = \vec{r} \times \vec{p}$ Fixed principal axis: $L_z = I_z \omega$
Equation of motion	$\sum \vec{F}_{\text{ext}} = \frac{d\vec{p}}{dt}$	External action changes momentum	$\sum \vec{\tau}_{\text{ext}, O} = \frac{d\vec{L}_O}{dt}$ Fixed axis, constant I_z : $\sum \tau_z = I_z \alpha$
Kinetic energy	$K_{\text{trans}} = \frac{1}{2} m v_{\text{CM}}^2$	Scalar measure of motion	$K_{\text{rot}} = \frac{1}{2} I_{\text{CM}} \omega^2$ Rolling rigid body: $K = \frac{1}{2} M v_{\text{CM}}^2 + \frac{1}{2} I_{\text{CM}} \omega^2$
Work and power	$W = \int \vec{F} \cdot d\vec{r}$ $P = \vec{F} \cdot \vec{v}$	Energy transfer	$W_{\text{rot}} = \int \tau_z d\theta$ $P_{\text{rot}} = \tau_z \omega$



Moment of inertia depends on the axis



Same mass $\neq$ same rotational inertia.
 I depends on mass distribution and the chosen axis.



KEY TAKEAWAY: Translation and rotation share a common mathematical structure—but rotational inertia and angular momentum depend on the chosen axis.

20. Conservation in Action: Figure Skater and Diver



Key takeaway: With zero (or negligible) net external torque, angular momentum is conserved. Changing body shape changes I , so ω adjusts.

$$I_1\omega_1 = I_2\omega_2$$

(about the stated rotation axis)

4.1 Figure Skater (spinning)

State 1: Arms extended



I_1 large ω_1 small

Net external torque
about the vertical axis
 ≈ 0 (ice friction negligible)



$$I_{z1} = I_{z2}$$

$$I_1\omega_1 = I_2\omega_2$$

State 2: Arms pulled in



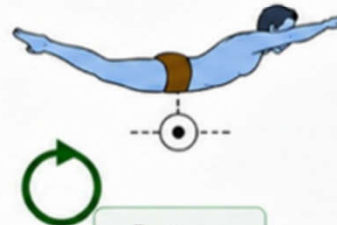
I_2 small ω_2 large

Example: $I_1 = 4.0 \text{ kg}\cdot\text{m}^2$, $\omega_1 = 2.0 \text{ rad/s}$, $I_2 = 1.6 \text{ kg}\cdot\text{m}^2$

$$\omega_2 = \frac{I_1\omega_1}{I_2} = \frac{4.0 \times 2.0}{1.6} = \mathbf{5.0 \text{ rad/s}}$$

4.2 Diver (in air)

State 1: Body extended



I_1 large
 ω_1 small

Net external torque
about the center of mass
 ≈ 0 (air resistance negligible)



$$L_{CM,1} = L_{CM,2}$$

$$I_1\omega_1 = I_2\omega_2$$

(about the same somersault axis)

State 2: Body tucked



I_2 small
 ω_2 large

Example: $I_1 = 10.0 \text{ kg}\cdot\text{m}^2$, $\omega_1 = 1.5 \text{ rad/s}$, $I_2 = 2.5 \text{ kg}\cdot\text{m}^2$

$$\omega_2 = \frac{I_1\omega_1}{I_2} = \frac{10.0 \times 1.5}{2.5} = \mathbf{6.0 \text{ rad/s}}$$



Big idea: Pulling mass closer to the axis decreases I and increases $|\omega|$.
Extending increases I and decreases $|\omega|$ — while angular momentum remains constant.

Important: Angular velocity is not conserved.
External torque or air resistance can change angular momentum.



Take-home message: When net external torque about the rotation axis is approximately zero, angular momentum is conserved: $I_1\omega_1 = I_2\omega_2$.



Next: Nature and Technology — Planetary Motion, Gyroscopes, and Spacecraft

21. Worked Example: Atwood Machine with a Massive Pulley



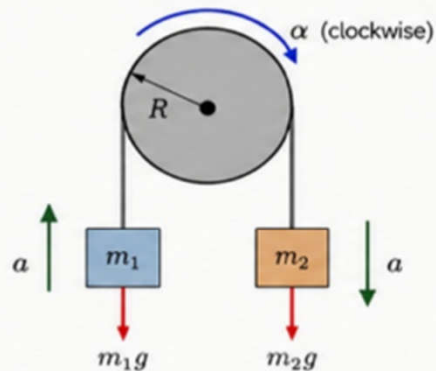
Key takeaway: Translation and rotation are coupled. Choose one consistent positive direction before writing the equations.

Sign convention

Positive directions:

m_1 upward, m_2 downward,
pulley clockwise.

Physical setup



Ideal massless rope, no slipping on pulley.

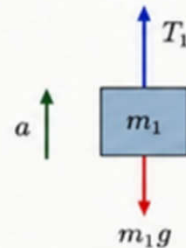
Pulley mass = M , moment of inertia = I .

Axle is frictionless.

Given: $m_2 > m_1$, so m_2 moves down and m_1 moves up with the same $|a|$.

Free-body diagrams: blocks

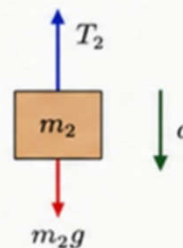
Block m_1 (left, accelerates up)



Newton's 2nd law (up positive):

$$T_1 - m_1g = m_1a$$

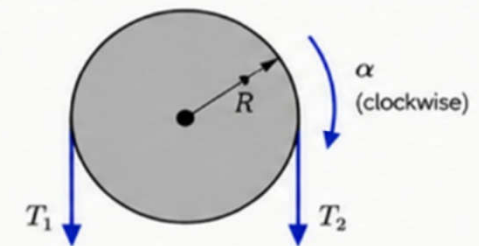
Block m_2 (right, accelerates down)



Newton's 2nd law (down positive):

$$m_2g - T_2 = m_2a$$

Free-body diagram: pulley



Torques about axle (clockwise positive):

$$(T_2 - T_1)R = I\alpha$$

Forces shown are exerted by the rope on the pulley.
(Axle force omitted; it produces zero torque.)

No-slip constraint

$$a = \alpha R$$

The rope's linear acceleration equals the tangential acceleration of the rim.

Final result

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{I}{R^2}}$$

Direction: m_2 downward and m_1 upward when $m_2 > m_1$.

Optional for homogeneous disk ($I = \frac{1}{2}MR^2$):

$$a = \frac{m_2 - m_1}{m_1 + m_2 + \frac{M}{2}}g$$

Numerical check

Given: $m_1 = 1.0$ kg, $m_2 = 3.0$ kg,
 $M = 2.0$ kg, $R = 0.10$ m,
 $I = \frac{1}{2}MR^2$, $g = 9.81$ m/s²

$$a = 3.92 \text{ m/s}^2 \quad (m_2 \text{ down, } m_1 \text{ up})$$

$$\alpha = 39.2 \text{ rad/s}^2 \quad (\text{clockwise})$$

$$T_1 = 13.7 \text{ N}$$

$$T_2 = 17.6 \text{ N}$$

Because the pulley has rotational inertia, the tensions are not equal: $T_1 \neq T_2$.
(Here, $T_2 > T_1$.)



Common mistake: Do not mix the force on a block with the equal-and-opposite force that the rope exerts on the pulley.

22. Engineering Applications of Rotational Motion

★ Key takeaway: Torque, moment of inertia, rotational energy, and angular momentum determine how rotating machines accelerate, store energy, transmit power, and maintain stability.

1. Flywheels – Energy Storage and Smoothing

$$E_{rot} = \frac{1}{2} I \omega^2$$

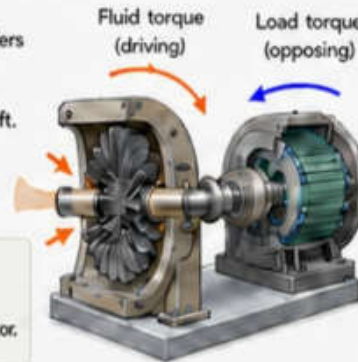
- Larger I or $|\omega|$ stores more energy.
- Flywheels smooth short-term speed and power fluctuations.



Stores rotational energy; does not create energy.

2. Turbines – Power Generation

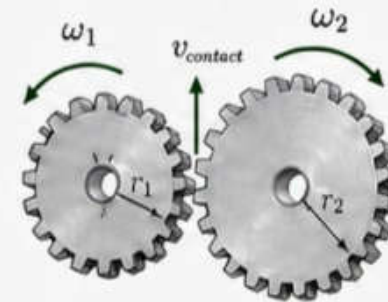
- Water, steam, or gas transfers energy to rotating blades.
- Fluid torque drives the shaft.
- Generator load provides opposing torque.



$$P = \tau \omega$$

Power delivered to the generator.

3. Gears – Speed and Torque Transmission



$$|\omega_1| r_1 = |\omega_2| r_2$$

$$\frac{|\omega_2|}{|\omega_1|} = \frac{r_1}{r_2}$$

For ideal gears:

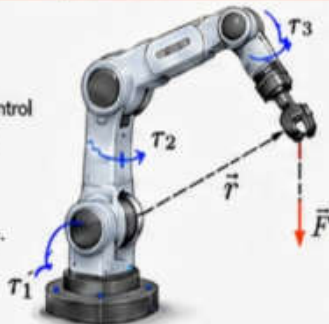
$$|\tau_1 \omega_1| = |\tau_2 \omega_2|$$

$$\frac{|\tau_2|}{|\tau_1|} = \frac{r_2}{r_1}$$

Ideal gears conserve power, not the angular momentum of each gear.

4. Robotics and Manipulators

- $\tau = r \times \vec{F}$ about a joint.
- Motor torque and feedback control joint acceleration and position.
- Payload farther from a joint requires more holding torque.



Increase in lever arm r increases required torque for the same force.

5. Satellites and Reaction Wheels

- Reaction wheel spins opposite to rotate the spacecraft body.
 - If net external torque ≈ 0 :
- $$\vec{L}_{body} + \vec{L}_{wheel} = \text{constant}$$
- Changing wheel speed reorients the spacecraft while conserving total system angular momentum.



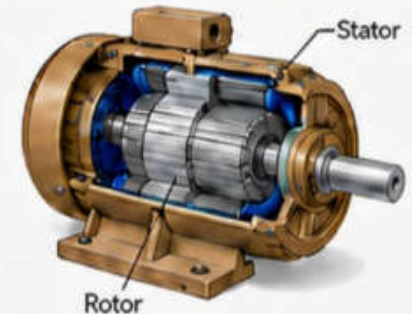
Applies to the total system (spacecraft + wheel) when external torque is negligible.

6. Electric Motors

- Magnetic forces produce torque on the rotor.

$$\tau_{net} = I \alpha$$

- Rotor inertia affects angular acceleration and transient response.



Electrical power input $\rightarrow$ torque $\rightarrow$ rotor rotation.

23. Common Mistakes in Circular and Rotational Motion

! Top takeaway: Choose the frame, axis, system, and sign convention before writing equations. Then check every vector direction and limiting case.

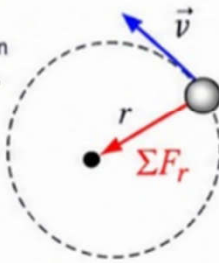
1 Centripetal and centrifugal force

✗ Mistake: Add a separate centripetal force, or add an outward centrifugal force in an inertial frame.

✓ Correct: In an inertial frame, the real forces have a net inward radial component.

$$\Sigma F_r = \frac{mv^2}{r} \quad (\text{positive inward})$$

A centrifugal force may be introduced only in a rotating non-inertial frame.



2 Torque magnitude and direction

✗ Mistake: Use $\tau = rF$ for every force or treat torque as a force.

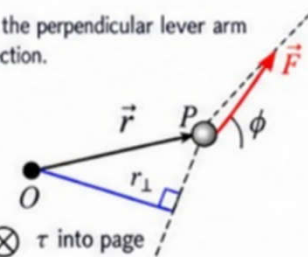
✓ Correct: Torque depends on the perpendicular lever arm and has an axial direction.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = rF \sin \phi = r_{\perp} F$$

$\odot$ τ out of page $\otimes$ τ into page

Reverse the cross-product order and the direction reverses.



3 Angular velocity and angular acceleration

✗ Mistake: Confuse ω with α .

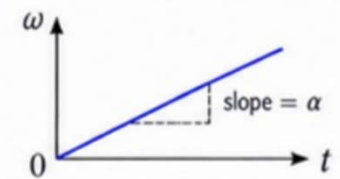
✓ Correct: ω describes how fast and in which sense θ changes; α describes how ω changes.

$$\omega = \frac{d\theta}{dt} \quad \alpha = \frac{d\omega}{dt}$$

Units:

$$[\omega] = \text{rad s}^{-1}$$

$$[\alpha] = \text{rad s}^{-2}$$



α is the slope of the ω - t graph.

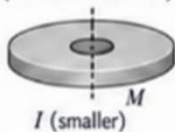
4 Moment of inertia

✗ Mistake: Assume I depends only on total mass.

✓ Correct: I depends on mass distribution relative to the chosen axis.

$$I = \int r_{\perp}^2 dm$$

Mass near axis
(same total mass M)



Mass farther from axis
(same total mass M)



Always specify the rotation axis.

5 Angular momentum

✗ Mistake: Compare angular momentum using ω alone.

✓ Correct: For fixed-axis rotation about a principal axis, $L_z = I_z \omega$. Both I_z and ω matter.

$$L_z = I_z \omega$$

In general,

$$\vec{L} = \mathbf{I} \cdot \vec{\omega}$$

$\mathbf{I}$ is the inertia tensor; $\vec{L}$ need not be parallel to $\vec{\omega}$.

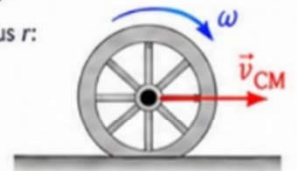
6 Linear relations and rotational energy

✗ Mistake: Mix linear and angular variables, or omit rotational kinetic energy.

✓ Correct: For a point at fixed radius r :

$$s = r\theta \quad v_t = r\omega$$

$$a_t = r\alpha \quad a_r = \omega^2 r$$



Rotational kinetic energy:

$$K_{\text{rot}} = \frac{1}{2} I \omega^2$$

For rolling without slipping:

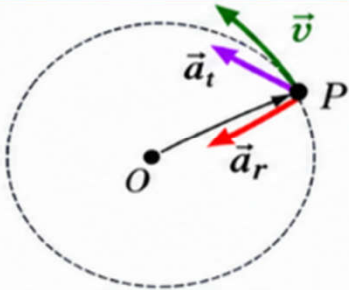
$$K_{\text{total}} = \frac{1}{2} M v_{\text{CM}}^2 + \frac{1}{2} I_{\text{CM}} \omega^2$$

$$v_{\text{CM}} = \omega R$$

24. Summary: Circular and Rotational Motion

★ **Key takeaway:** Geometry connects linear and angular motion; torque changes angular momentum; energy provides an independent solution method.

1 Circular kinematics



For circular motion with fixed radius r :

$$\begin{aligned}s &= r\theta \\ v_t &= r\omega \\ a_t &= r\alpha \\ a_r &= \frac{v^2}{r} = \omega^2 r\end{aligned}$$

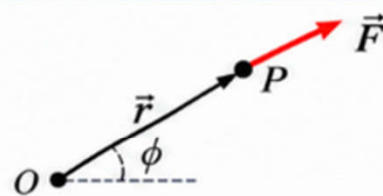
Vector radial relation:

$$\vec{a}_r = -\omega^2 \vec{r}$$

(directed toward the center)

i a_r changes direction;
 a_t changes the speed.

2 Torque and moment of inertia



Torque (about O):

$$\begin{aligned}\vec{\tau} &= \vec{r} \times \vec{F} \\ |\vec{\tau}| &= rF \sin \phi = r_{\perp} F \\ r_{\perp} &= r \sin \phi\end{aligned}$$

Moment of inertia:

$$I = \int r_{\perp}^2 dm$$

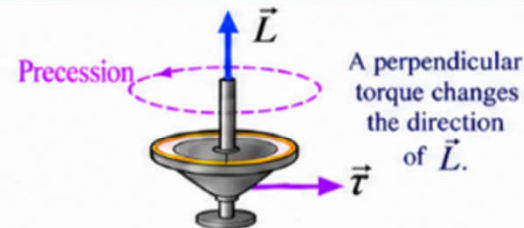
Fixed axis, constant I : $\sum \tau_z = I_z \alpha$

Parallel-axis theorem:

$$I = I_{\text{CM}} + Md^2$$

i Only the component of $\vec{F}$ perpendicular to $\vec{r}$ produces torque.

3 Angular momentum and conservation



Angular momentum (about O):

$$\vec{L}_O = \vec{r} \times \vec{p}$$

Fixed principal axis:

$$L_z = I_z \omega_z$$

General law (about O):

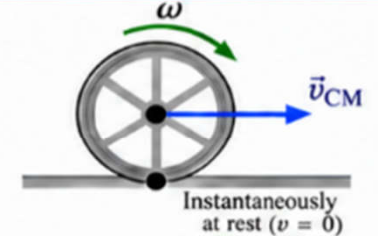
$$\sum \vec{\tau}_{\text{ext}, O} = \frac{d\vec{L}_O}{dt}$$

If $\sum \vec{\tau}_{\text{ext}, O} = 0$:

$$\vec{L}_O = \text{constant}$$

i In general, $\vec{L} = \mathbf{I} \cdot \vec{\omega}$;
 $\vec{L}$ need not be parallel to $\vec{\omega}$.

4 Rotational energy and rolling



Rotational kinetic energy:

$$K_{\text{rot}} = \frac{1}{2} I \omega^2$$

Rolling without slipping:

$$v_{\text{CM}} = \omega R \quad a_{\text{CM}, t} = \alpha R$$

Total kinetic energy:

$$K_{\text{total}} = \frac{1}{2} M v_{\text{CM}}^2 + \frac{1}{2} I_{\text{CM}} \omega^2$$

Work-energy: $W_{\text{ext}} = \Delta K_{\text{total}}$

Power: $P = \vec{\tau} \cdot \vec{\omega}$

i For ideal rolling on a stationary rigid surface, static friction has zero instantaneous power at the contact point.

Problem-solving workflow

- 1 Define the system and reference frame. →
- 2 Choose the axis, origin, and positive directions. →
- 3 Draw the free-body diagram and identify constraints. →
- 4 Apply $\sum \vec{F} = m\vec{a}$, $\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$, or an energy method. →
- 5 Check units, signs, limiting cases, and vector directions.

★ **Take-home message:** Clear diagrams, consistent signs, correct assumptions, and explicit validity conditions lead to reliable solutions.