

PHYSICS FOR ENGINEERING STUDENTS

Thin Lenses and Lens Systems

Image Formation and Optical Design

Lecture 7 — Book-style student notes

Learning focus

Refraction and focal length

Thin-lens equation and magnification

Principal-ray construction

Real and virtual images

Compound and matrix lens systems

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1. Why lenses are needed

From collecting light to forming a sharp image

An optical image is not produced merely by allowing light to reach a screen. Every point of an extended object sends rays in many directions. If a large opening is placed in front of a screen without any focusing element, rays originating from different object points overlap. The screen receives plenty of light, but it cannot preserve the spatial correspondence between object points and image points; the result is bright and blurred. A pinhole improves the mapping because it admits only a narrow bundle from each point, but the improvement is purchased at the cost of very low brightness and diffraction-limited detail.

A lens solves this compromise. It accepts a comparatively wide cone of rays from each object point and changes their directions so that the rays meet, or appear to meet, at one corresponding image point. Thus a lens can form an image that is both bright and sharp. This point-to-point mapping is the central physical idea behind cameras, the human eye, microscopes, telescopes and projectors.

Refraction as the physical mechanism

A lens controls rays through refraction at two curved interfaces. When light enters a medium with a different refractive index, its speed changes and its direction usually changes as well. Snell's law relates the angles measured from the surface normal:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

If $n_2 > n_1$, the ray bends toward the normal; if $n_2 < n_1$, it bends away. A curved surface presents a different normal to rays at different heights. Consequently, each ray receives a different angular deflection. Two carefully shaped surfaces can make initially parallel rays converge to one point. The thin-lens model idealizes both refractions as occurring in a single plane, which is accurate when lens thickness is small compared with the relevant object, image and curvature distances.

A useful problem-solving habit

Before writing an equation, identify the optical task: What rays enter? Must they actually meet, or only appear to originate from a point? Where could a screen be placed? This physical sketch prevents sign errors later. Equations encode the geometry, but they do not decide the geometry for us.

Arrangement	Light level	Image quality
Pinhole	Low	Reasonably localized, diffraction-limited
Large aperture without lens	High	Blurred because bundles overlap
Large aperture with lens	High	Sharp if the screen is at the image plane

2. Lens types, axes and focal points

The geometrical vocabulary of thin-lens optics

A convex lens is thicker near its center than at its rim. In air it is normally converging: rays parallel to the principal axis are brought to a real focus on the far side, and its focal length is positive. A concave lens is thinner near the center and normally diverging: parallel rays spread after refraction, but their backward extensions intersect at a virtual focus on the incident side. Its focal length is negative.

Reference elements

The principal axis is the straight reference line through the lens center and perpendicular to the idealized lens plane. The optical center O is the reference point from which object distance, image distance and focal length are measured. In the thin-lens approximation, a ray passing through O continues essentially undeviated. Each lens has two focal points, one on each side, symmetrically positioned at distance $|f|$ from O when the medium is the same on both sides.

Quantity	Meaning	Sign in this lecture
f	Focal length; strength and lens type	$f > 0$ converging; $f < 0$ diverging
d_o	Object distance from the lens	Real object on incident side: positive
d_i	Image distance from the lens	Real image on outgoing side: positive
h_o, h_i	Object and image height	Above axis positive; below axis negative

Focal length and optical power

The focal length measures how rapidly the lens changes ray direction. A short $|f|$ corresponds to strong bending; a long $|f|$ corresponds to weak bending. Optical power is the reciprocal of focal length measured in metres:

$$P = 1/f \quad (P \text{ in dioptres, } f \text{ in metres})$$

For example, $f = +0.20$ m gives $P = +5.0$ D, whereas $f = -0.50$ m gives $P = -2.0$ D. Power is especially useful for comparing or combining lenses because powers add for thin lenses in contact. The sign is not decoration: it states whether the element converges or diverges a parallel bundle.

Engineering interpretation

Focal length determines much more than “zoom.” For a given sensor size, a shorter camera focal length produces a wider field of view, while a longer focal length gives a narrower field and a larger image of a distant subject. Microscope objectives use short focal lengths to obtain strong angular changes. Telescope magnification depends on the ratio of objective and eyepiece focal lengths. Projector focal length helps determine image size at a specified throw distance. In every case, f links angles in object space to positions in image space.

3. The thin-lens equation

Predicting where an image will form

The thin-lens equation connects the focal length with the object and image distances. With the sign convention summarized above, it is written as

$$1/f = 1/d_o + 1/d_i$$

It may be rearranged to solve directly for the image distance:

$$d_i = f d_o / (d_o - f)$$

This form makes several limits transparent. When d_o is very large, d_i approaches f : distant objects form images near the focal plane. When $d_o = f$, the denominator vanishes and the outgoing rays are parallel, so the image is at infinity. When a converging lens has $d_o < f$, the denominator is negative and $d_i < 0$; the image is virtual and lies on the object side.

Systematic solution protocol

- Draw the axis, lens, object and focal points; make a qualitative prediction.
- List the known quantities with units and assign signs before substitution.
- Choose the form of the lens equation that isolates the unknown.
- Calculate while retaining units and at least one extra significant figure.
- Interpret the sign of d_i and compare the result with the ray sketch.

Worked calculation: object beyond twice the focal length

A converging lens has $f = +10.0$ cm and an object is placed 30.0 cm in front of it. Substitution gives

$$\begin{aligned} 1/d_i &= 1/(10.0 \text{ cm}) - 1/(30.0 \text{ cm}) = 2/(30.0 \text{ cm}) \\ d_i &= +15.0 \text{ cm} \end{aligned}$$

The positive result means that the rays actually meet 15.0 cm behind the lens. Because $f < d_i < 2f$, the image lies between F and $2F$. This agrees with the qualitative rule for an object beyond $2F$: the image must be real, inverted and reduced.

Dimensional and physical checks

All distances in one calculation must use the same unit. The reciprocal equation has dimensions of inverse length, so mixing centimetres and metres without conversion is an immediate error. A second check is limiting behavior: a very distant object cannot produce an image much farther than f from a positive lens. A third check is screen test: only $d_i > 0$ corresponds to an image that can be projected onto a screen in the standard real-object arrangement.

4. Magnification and image orientation

Connecting image distance to image size

Transverse magnification compares image height with object height. Similar triangles in the ray diagram give two equivalent expressions:

$$m = h_i/h_o = -d_i/d_o$$

The magnitude $|m|$ gives the size ratio. If $|m| > 1$, the image is enlarged; if $|m| < 1$, it is reduced; and if $|m| = 1$, object and image have equal height. The sign describes orientation. Negative magnification means that the image is inverted relative to the object, while positive magnification means that it is upright.

Worked continuation

For the previous example, $d_o = 30.0$ cm and $d_i = 15.0$ cm. Therefore

$$\begin{aligned} m &= -15.0/30.0 = -0.500 \\ \text{If } h_o &= +4.0 \text{ cm, then } h_i = m h_o = (-0.500)(4.0 \text{ cm}) = -2.0 \text{ cm} \end{aligned}$$

The image is half as tall as the object and inverted. Notice that “negative height” does not mean a physically impossible size. Height magnitude is positive; the minus sign records that the top of the object maps below the axis.

Combined algebraic form

Eliminating d_i between the lens and magnification equations yields a form useful when f and d_o are known:

$$m = f/(f - d_o)$$

This formula should not replace reasoning. For example, if $f > 0$ and $d_o > f$, the denominator is negative, so $m < 0$ and a real image is inverted. If $0 < d_o < f$, the denominator is positive, so $m > 0$ and the virtual image is upright. The algebra reflects the ray geometry.

m	Orientation	Relative size
$m < -1$	Inverted	Enlarged
$-1 < m < 0$	Inverted	Reduced
$m = -1$	Inverted	Same size
$0 < m < 1$	Upright	Reduced
$m > 1$	Upright	Enlarged

Common interpretation error

A convex lens does not always magnify. Its behavior depends on object position. Beyond $2F$ it produces a smaller real image; at $2F$ the image has equal size; between F and $2F$ it produces a larger real image; and inside F it produces a larger virtual image. “Convex” describes ray bending, not a single fixed image size.

5. The three principal rays

A geometrical method for locating images

A full optical bundle contains infinitely many rays, yet a thin-lens image can be located with only two carefully selected rays. A third ray provides a consistency check. These principal rays are not special physical rays; they are convenient members of the bundle whose paths are known from focal geometry.

Ray 1: parallel ray

A ray incident parallel to the principal axis leaves a converging lens through the far focal point. At a diverging lens it leaves in a direction whose backward extension passes through the near focal point. This rule explains why light from a distant object is concentrated near the focal plane.

Ray 2: optical-center ray

A ray directed through the optical center continues in a straight line in the thin-lens approximation. The two surface deviations approximately cancel because the lens is treated as having negligible thickness. This ray gives a stable diagonal reference through object and image tips.

Ray 3: focal ray

A ray that approaches a converging lens through the near focal point emerges parallel to the axis. The rule is the reversible counterpart of Ray 1. For a diverging lens, a ray aimed toward the far focal point emerges parallel. Reversibility is an important optical principle: if a permitted path is traversed backward, it remains a permitted path.

Construction procedure

- Draw the principal axis and represent the lens by a vertical line through O.
- Mark F and F' at equal distances $|f|$ on both sides.
- Draw the object as an upright arrow and start rays from its tip.
- Construct any two principal rays with a ruler.
- Their intersection is the image tip; if refracted rays diverge, extend them backward with dashed lines.
- Drop a perpendicular to the axis to complete the image arrow.

If the rays fail to meet at one point, the drawing—not the laws of optics—is inconsistent. Recheck focal-point placement, ensure that every ray begins at the same object point, and apply the correct converging or diverging rule. A well-made ray diagram predicts image side, orientation and approximate size before calculation, making it one of the best error-detection tools in lens problems.

6. Image formation by a converging lens

Five object positions that organize all cases

The behavior of a positive lens is classified by comparing the object distance d_o with f and $2f$. The borders are physically meaningful: F separates real from virtual imaging, while $2F$ marks unit magnification.

Object position	Image position	Nature	Size
$d_o > 2f$	Between F' and 2F'	Real, inverted	Reduced
$d_o = 2f$	At 2F'	Real, inverted	Same
$f < d_o < 2f$	Beyond 2F'	Real, inverted	Enlarged
$d_o = f$	At infinity	No finite screen image	—
$0 < d_o < f$	Object side	Virtual, upright	Enlarged

How the image moves

Imagine moving an object from very far away toward the lens. Its real image begins close to F' and moves away from the lens while growing. At $d_o = 2f$ the image reaches $2F'$ and has equal size. As the object approaches F from outside, d_i and m increase without bound. Exactly at F , the output is collimated. Moving the object just inside F changes the topology: the refracted rays diverge and their backward extensions define a very distant, upright virtual image on the object side. As the object approaches the lens, that virtual image approaches the lens and its magnification decreases toward one.

Real versus virtual

A real image is located where rays physically converge. It can be intercepted by a screen because light energy passes through every point of the image plane. A virtual image is located where backward extensions converge; no light actually passes through that apparent point, so placing a screen there does not produce the image. Nevertheless, an eye or camera can receive the diverging rays and trace them backward, which is why virtual images are visible.

Applications

A camera normally operates with $d_o > f$ and places its sensor at a real image plane. A projector places a small object between F and $2F$ to produce an enlarged real image on a distant screen. A simple magnifier places the observed object inside F , producing an upright enlarged virtual image. The same lens can serve all three roles; object position selects the operating regime.

When solving a problem, first choose the correct row of the table. Then use the equations to obtain numerical values. If the signs and magnitudes disagree with the selected row, revisit the sign assignment or arithmetic.

7. Diverging lenses

Virtual, upright and reduced images

For a real object, an isolated diverging lens always produces a virtual, upright and reduced image between the lens and its near focal point. The focal length is negative. The same thin-lens and magnification equations remain valid; no special replacement formula is needed.

Worked example

Let $f = -20.0$ cm, $d_o = +40.0$ cm and $h_o = +3.0$ cm. Then

$$1/d_i = 1/(-20.0 \text{ cm}) - 1/(40.0 \text{ cm}) = -3/(40.0 \text{ cm})$$

$$d_i = -13.33 \text{ cm}$$

$$m = -d_i/d_o = -(-13.33)/40.0 = +0.333$$

$$h_i = m h_o = (+0.333)(3.0 \text{ cm}) = +1.00 \text{ cm}$$

The negative image distance places the image on the object side, hence it is virtual. Positive m means upright, and $|m| < 1$ means reduced. All three conclusions agree with the principal-ray diagram: the outgoing rays diverge, while their backward extensions meet between O and F.

Limiting cases

As d_o tends to infinity, the image approaches the near focal point and becomes extremely small. As the object approaches the lens, the image also approaches the lens and $|m|$ approaches one from below. A diverging lens cannot form a real image from an ordinary real object by itself, but it can contribute to a real final image in a compound system because the object presented to it may be virtual.

Engineering uses

Negative lenses correct myopia by reducing the convergence of rays before they enter the eye. Door viewers use a diverging group to create a wide field of view with a reduced virtual image. Galilean telescopes and beam expanders combine positive and negative elements to control angular magnification or beam diameter. These examples emphasize why lens type alone is not the whole system: the sequence and separation of elements matter.

Quick check

For a standard real object and $f < 0$, a result with $d_i > 0$ or $m < 0$ should trigger suspicion. It may indicate inconsistent signs. Always compare the calculated result with the invariant qualitative expectation: virtual, upright and smaller.

8. Worked examples with a converging lens

From prediction to calculation and interpretation

Example A: object at 2F

A converging lens has $f = +15.0$ cm. An object of height $h_o = +2.0$ cm is at $d_o = +30.0$ cm. Prediction: because $d_o = 2f$, the image should be real, inverted, equal in size and located at 2F on the other side.

$$\begin{aligned} 1/d_i &= 1/15.0 - 1/30.0 = 1/30.0 \text{ cm}^{-1} \Rightarrow d_i = +30.0 \text{ cm} \\ m &= -30.0/30.0 = -1.00; \quad h_i = (-1.00)(2.0 \text{ cm}) = -2.0 \text{ cm} \end{aligned}$$

The signs confirm the prediction. A screen placed 30.0 cm behind the lens receives an image 2.0 cm high, inverted relative to the object.

Example B: object between F and 2F

Keep $f = +15.0$ cm and $h_o = +2.0$ cm, but move the object to $d_o = +20.0$ cm. Prediction: a real, inverted and enlarged image beyond 2F.

$$\begin{aligned} 1/d_i &= 1/15.0 - 1/20.0 = 1/60.0 \text{ cm}^{-1} \Rightarrow d_i = +60.0 \text{ cm} \\ m &= -60.0/20.0 = -3.00; \quad h_i = (-3.00)(2.0 \text{ cm}) = -6.0 \text{ cm} \end{aligned}$$

The image is three times taller and forms four focal lengths from the lens. This illustrates the rapid change in image position as the object approaches F from outside.

Example C: object inside F

Now place the same object at $d_o = +10.0$ cm.

$$\begin{aligned} 1/d_i &= 1/15.0 - 1/10.0 = -1/30.0 \text{ cm}^{-1} \Rightarrow d_i = -30.0 \text{ cm} \\ m &= -(-30.0)/10.0 = +3.00; \quad h_i = +6.0 \text{ cm} \end{aligned}$$

The negative image distance and positive magnification describe a virtual, upright, enlarged image. It cannot be caught on a screen at $d_i = -30.0$ cm, because the rays do not actually travel to that location.

General lesson

Prediction is not an optional afterthought. It supplies an independent answer category—real or virtual, upright or inverted, enlarged or reduced—against which the arithmetic can be tested. A correct numerical calculation must fit the correct physical category.

9. Two lenses in contact

Equivalent focal length and optical power

Two thin lenses in contact may be replaced by a single equivalent thin lens. Because there is negligible propagation distance between the elements, their optical powers add:

$$1/f_{eq} = 1/f_1 + 1/f_2 \quad \text{or} \quad P_{eq} = P_1 + P_2$$

The rule applies to any signed combination. If the reciprocal sum is positive, the group is converging; if negative, it is diverging; if zero, it is afocal and has infinite equivalent focal length. “In contact” is essential: appreciable separation introduces an additional term and must be handled sequentially or by matrices.

Examples

f_1	f_2	$1/f_{eq}$	f_{eq}	Result
+10 cm	+20 cm	$3/20 \text{ cm}^{-1}$	+6.67 cm	Stronger converging
+10 cm	−20 cm	$1/20 \text{ cm}^{-1}$	+20.0 cm	Weak converging
+10 cm	−10 cm	0	∞	Afocal
−15 cm	−30 cm	$-1/10 \text{ cm}^{-1}$	−10.0 cm	Diverging

Worked combination

A +10 cm lens touches a −20 cm lens. The equivalent focal length is +20 cm. If an object is 60 cm in front of the combination, then

$$1/d_i = 1/20 - 1/60 = 1/30 \text{ cm}^{-1} \Rightarrow d_i = +30 \text{ cm}$$

$$m = -30/60 = -0.50$$

Thus the combination forms a real, inverted image half the object height. Treating only the +10 cm element would predict a much closer image and would ignore the power canceled by the negative lens.

Design interpretation

Real objectives rarely use one element. Designers combine positive and negative lenses to obtain the required net power while also correcting aberrations, controlling field curvature and setting mechanical distances. The equivalent-power calculation is therefore a first-order design tool, not a complete description of image quality. It predicts paraxial focus but does not describe chromatic or off-axis errors.

Error prevention

Convert every focal length to the same unit before taking reciprocals. Preserve signs through the entire reciprocal calculation. Finally, estimate the answer: adding a positive lens should increase positive power unless a negative element cancels it. If two equal and opposite powers are combined, a finite f_{eq} is impossible.

10. Separated lenses

The intermediate image becomes the next object

When lenses are separated, the first lens forms an intermediate image. That image—whether real or virtual—serves as the object for the second lens. The safest method is sequential: solve one lens, transfer the geometry, then solve the next. Distances must be measured from the optical center of the lens currently being analyzed.

General two-lens algorithm

- Use $1/f_1 = 1/d_{o1} + 1/d_{i1}$ to find the first image.
- Locate that image on a common axis relative to lens 1.
- If lens 2 is distance L to the right, compute its object distance from the relative positions. With a left-to-right real-object convention, $d_{o2} = L - d_{i1}$ when the intermediate image is to the left of lens 2.
- Apply the lens equation to lens 2.
- Compute m_1 and m_2 ; the total transverse magnification is $m = m_1 m_2$.
- Interpret the final image relative to the original object, not merely relative to the intermediate image.

Worked two-lens system

Lens 1 has $f_1 = +10$ cm, lens 2 has $f_2 = +15$ cm, and their separation is $L = 20$ cm. An object lies 30 cm left of lens 1 with height 2.0 cm.

$$\text{Lens 1: } 1/d_{i1} = 1/10 - 1/30 \Rightarrow d_{i1} = +15 \text{ cm}; \quad m_1 = -15/30 = -0.50$$

The intermediate image is 15 cm to the right of lens 1, hence 5 cm to the left of lens 2. It is a real object for lens 2, so $d_{o2} = +5$ cm.

$$\text{Lens 2: } 1/d_{i2} = 1/15 - 1/5 = -2/15 \Rightarrow d_{i2} = -7.5 \text{ cm}; \quad m_2 = -(-7.5)/5 = +1.50$$

$$m = m_1 m_2 = (-0.50)(+1.50) = -0.75; \quad h_{\text{final}} = -1.50 \text{ cm}$$

The final image is virtual with respect to lens 2, 7.5 cm to its left. Relative to the original object it is inverted and reduced. The product rule for magnification is essential: the second lens is upright relative to the already inverted intermediate image, so the final image remains inverted relative to the original.

A frequent mistake

Do not reuse d_{i1} directly as d_{o2} . The numerical distance for lens 2 depends on the separation and on which side of lens 2 the intermediate image lies. A one-dimensional position sketch eliminates this ambiguity.

11. Multiple lenses and ABCD matrices

A systematic framework for paraxial ray tracing

Sequential lens equations are intuitive, but many-element systems become cumbersome. Paraxial matrix optics represents a ray by its height y and small angle θ . Translation through free space and refraction by a thin lens are linear transformations:

$$\text{Ray vector } r = [y; \theta]$$

$$\text{Translation } T(L) = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \quad \text{Thin lens } Q(f) = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

For a column-vector convention, transformations act in the order encountered by the ray. A system of lens 1, free space L , then lens 2 has

$$M = Q(f_2) T(L) Q(f_1) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The output vector is $r_{\text{out}} = M r_{\text{in}}$. Matrix multiplication automatically carries ray height and angle through every element and is therefore well suited to software, optimization and tolerance analysis. The determinant of each ideal translation and thin-lens matrix is one, so the system determinant is also one in the same refractive index; this is a useful algebraic check.

Physical meaning of the system matrix

Element	Interpretation
A	How input height contributes to output height
B	How input angle contributes to output height
C	How input height contributes to output angle; related to optical power
D	How input angle contributes to output angle

An imaging condition is obtained when all rays from one object point arrive at one image point independent of their initial angle. In an object-to-image system matrix, this requires $B = 0$. An afocal system has $C = 0$: parallel input rays remain parallel, although their height and angle may be scaled.

Why both methods matter

The step-by-step lens equation makes intermediate images visible and supports physical intuition. The matrix method is compact and less error-prone for many elements, but its multiplication order and sign conventions must be defined consistently. The two methods should give the same final paraxial image. Using both on a small test system is an excellent validation strategy before trusting a larger calculation.

Scope of the approximation

ABCD matrices describe first-order, small-angle optics. They do not by themselves predict spherical aberration, coma, astigmatism, distortion, diffraction or wavelength dependence. They are the foundation of system layout; detailed optical design adds surface curvature, glass dispersion, apertures and numerical ray tracing.

12. Problem-solving checklist and common mistakes

A reliable workflow for examination and design problems

Unified workflow

- Sketch the axis, propagation direction, lenses, focal points and object.
- State the sign convention explicitly; never infer a sign from an unsigned diagram label.
- Predict image category before calculation.
- For one lens, solve the thin-lens equation first and magnification second.
- For several lenses, locate each intermediate image in a common coordinate system.
- Check units, significant figures, limiting behavior and screen feasibility.
- Use a principal-ray construction or an independent matrix calculation as verification.

Typical misunderstandings

Mistake	Why it fails	Correction
“The focal point is the image.”	Only an on-axis parallel bundle focuses at F .	Trace rays from the actual object point.
“A convex lens always enlarges.”	Magnification depends on d_o/f .	Classify the object relative to F and $2F$.
Negative m means negative size.	The sign encodes orientation.	Use $ m $ for size ratio and sign for upright/inverted.
A virtual image does not exist.	It is a valid apparent origin of rays.	Test whether rays or backward extensions meet.
Use the first image distance as the second object distance.	Distances are referenced to different lenses.	Convert positions using the lens separation.

Final conceptual synthesis

A lens forms images because curved refracting surfaces impose a controlled relation between ray height and direction. Focal length summarizes the first-order strength of that relation. The thin-lens equation converts object position into image position; magnification converts position geometry into size and orientation. Principal rays provide a visual construction, while sequential equations and ABCD matrices extend the same ideas to compound systems.

Good optical problem solving alternates between three representations: a physical sketch, an algebraic model and a verbal interpretation. None is sufficient alone. The sketch reveals geometry, the equations provide precision, and the interpretation tests whether the result could occur. When these three agree, the solution is usually reliable.

Core formula sheet

$$\text{Snell: } n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad \text{Lens: } 1/f = 1/d_o + 1/d_i$$

$$\text{Magnification: } m = h_i/h_o = -d_i/d_o \quad \text{Power: } P = 1/f$$

$$\text{Contact lenses: } 1/f_{eq} = \Sigma(1/f_j) \quad \text{Compound magnification: } m = \Pi m_j$$

The final discipline is to attach meaning to every sign. Positive and negative values identify lens type, image side and orientation. Once the geometry is encoded consistently, the equations become concise and powerful tools rather than sources of confusion.