

Kinematics

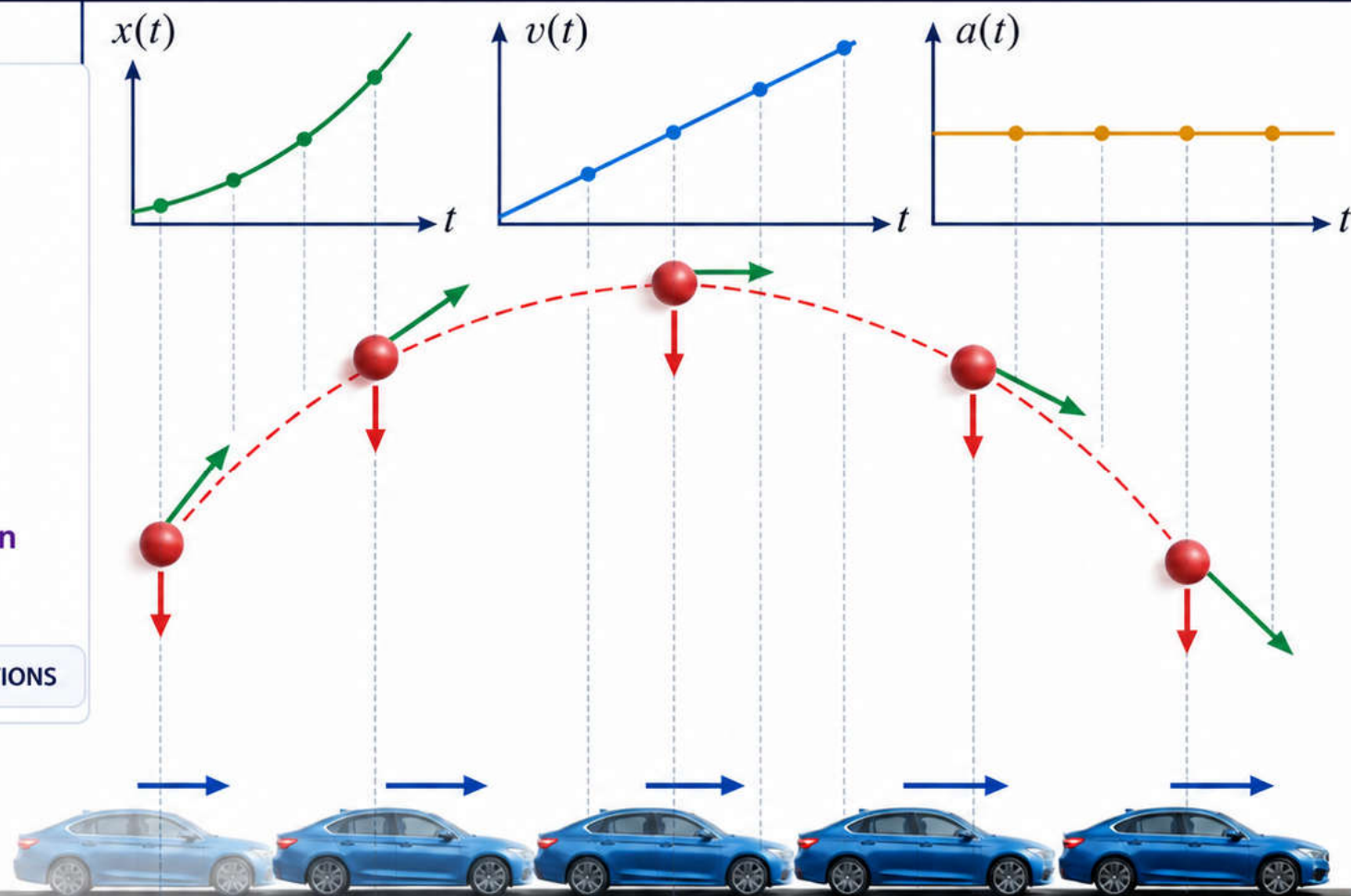
DESCRIBING MOTION

From Position to Prediction

Position → Velocity → Acceleration

A visual problem-solving introduction

ONE MOTION — THREE COMPLEMENTARY DESCRIPTIONS

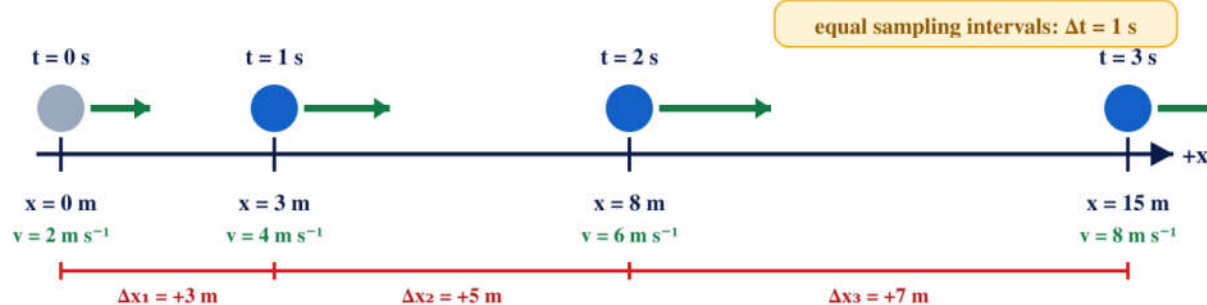


Predicting motion means determining $x(t)$ and $v(t)$

A complete prediction combines an initial state with a valid evolution law.

MOTION INFOGRAPHIC — equal times, increasing displacements

Example: $x_0 = 0 \text{ m}$, $v_0 = 2 \text{ m s}^{-1}$, $a = +2 \text{ m s}^{-2}$ (constant)



WHY DO WE NEED THIS?

Engineering questions concern a future state:
Where will the object be? How fast will it move?
A sketch alone cannot give a reproducible prediction.

THE MODEL: STATE + RULE → PREDICTION

initial state + evolution law → $x(t)$, $v(t)$

Here the rule is $a = \text{constant}$. Therefore velocity changes linearly with time, while position changes quadratically.

EQUATIONS — PHYSICAL MEANING OF EACH TERM

- $v = v_0 + at$
 v_0 : velocity at $t = 0$; at : change produced by a during t ; their sum gives the velocity v at time t .
- $x = x_0 + v_0t + \frac{1}{2}at^2$
 x_0 : position at $t = 0$; v_0t : displacement from initial motion; $\frac{1}{2}at^2$: contribution of acceleration.
- $v^2 = v_0^2 + 2a(x - x_0)$
 v_0^2 : initial state; $2a\Delta x$: change in v^2 produced over displacement Δx ; their sum gives v^2 .
- $x - x_0 = \frac{v_0 + v}{2}t$
 $\frac{1}{2}(v_0 + v)$: mean velocity for constant a ; multiplying it by t gives the displacement Δx .

WHAT EACH SYMBOL MEANS

x_0 initial position [m]
 x position at time t [m]
 v_0 initial velocity [m s⁻¹]
 v velocity at time t [m s⁻¹]
 a constant acceleration [m s⁻²]
 t elapsed time, $t \geq 0$ [s]

RESULT AND LIMIT

For the example at $t = 3 \text{ s}$:
 $x(3) = 15 \text{ m}$
 $v(3) = 8 \text{ m s}^{-1}$
 Growing 3–5–7 m gaps show $a > 0$.
 Use these equations only when acceleration is constant.

Kinematics describes how an object moves

Motion is described before we identify the interactions that cause it.

FROM OBSERVATION TO DESCRIPTION

Record position as a function of time. Derivatives describe local motion.



POSITION $x(t)$

$\frac{dx}{dt}$

VELOCITY $v(t)$

$\frac{dv}{dt}$

ACCELERATION $a(t)$

WHY SEPARATE DESCRIPTION FROM CAUSE?

The same equations describe a vehicle, a falling object, or a charged particle. Forces explain why their motion changes.

KINEMATICS: HOW?

$x(t)$, $v(t)$, $a(t)$

DYNAMICS: WHY?

forces, mass, interactions

MATHEMATICAL RELATIONS AND PHYSICAL CONTENT

- 1 $v(t) = \frac{dx(t)}{dt}$ Position changes; its derivative gives the instantaneous velocity.
- 2 $a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2}$ Velocity changes; the second derivative of position gives acceleration.
- 3 $x(t) - x(0) = \int_0^t v(\tau) d\tau$ Integrating velocity gives the change in position between 0 and t.
- 4 $v(t) - v(0) = \int_0^t a(\tau) d\tau$ Integrating acceleration gives the change in velocity between 0 and t.

FUNCTIONS AND MEANING

- $x(t)$ — position at time t
- $v(t)$ — how quickly position changes
- $a(t)$ — how quickly velocity changes

RESULT AND LIMIT

Knowing $x(t)$ gives $v(t)$ and $a(t)$.

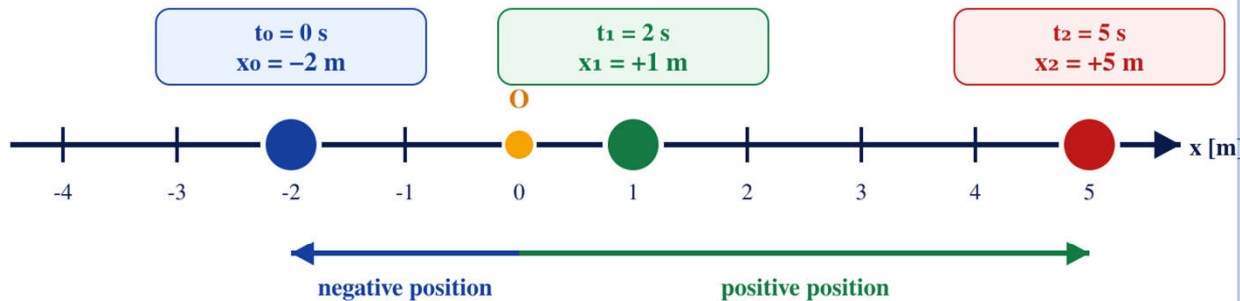
This describes motion; explaining its physical cause requires dynamics.

Position tells us where the object is

A position is a signed coordinate assigned at a specified time in a specified reference frame.

INFOGRAPHIC — ONE ROAD, THREE OBSERVATIONS

Reference frame: roadside; origin O at the gate; positive x points to the right.



WHY DEFINE POSITION FIRST?

Motion means that position changes with time. Without $x(t)$, displacement, velocity and acceleration cannot be defined quantitatively.

WHAT MUST ACCOMPANY A POSITION VALUE?

1. the reference frame and its origin O;
2. the positive direction and coordinate axis;
3. the time t at which the position is reported.

POSITION EQUATIONS — PHYSICAL MEANING

1 $x = x(t)$

x : signed coordinate measured from O; t : clock reading;
 $x(t)$: the object's position assigned at that instant.

2 $\vec{r}(t) = x(t) \hat{i}$

$x(t)$: signed scalar component; $\hat{i}$: fixed unit vector that specifies the positive axis; $\vec{r}(t)$: position vector.

Example: at $t_1 = 2$ s, $x(t_1) = +1$ m — one metre to the right of O.

HOW TO INTERPRET THE SIGN

$x < 0$ — object is left of O

$x = 0$ — object is at O

$x > 0$ — object is right of O

The sign states location relative to O—not the direction of motion.

POSITION $\neq$ DISTANCE

Position is a coordinate and may be negative, zero or positive. Distance travelled accumulates along the path and is never negative. A single x value does not reveal how the object arrived there.

Displacement: change of position and its sign

Subtract the initial coordinate from the final coordinate on a specified axis.

ONE COORDINATE AXIS · TWO MOTIONS

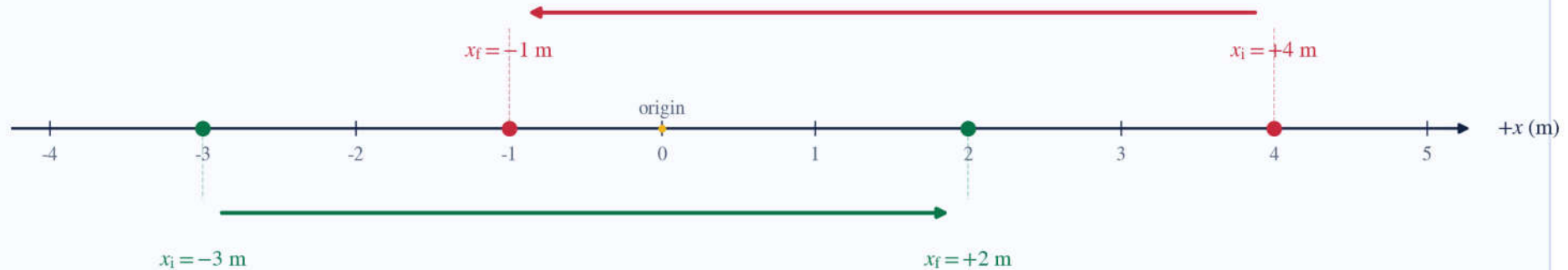
$$\Delta x = x_f - x_i$$

MOTION 1

toward $-x$

MOTION 2

toward $+x$



The arrow points from the initial position to the final position.

MOTION 1 · NEGATIVE

$$\Delta x = (-1 \text{ m}) - (+4 \text{ m})$$

$$\Delta x = -5 \text{ m}$$

Final coordinate is 5 m to the left of the start.

MOTION 2 · POSITIVE

$$\Delta x = (+2 \text{ m}) - (-3 \text{ m})$$

$$\Delta x = +5 \text{ m}$$

Final coordinate is 5 m to the right of the start.

WHAT THE SIGN MEANS

$\Delta x > 0$: toward $+x$

$\Delta x < 0$: toward $-x$

Only the two endpoints determine displacement.

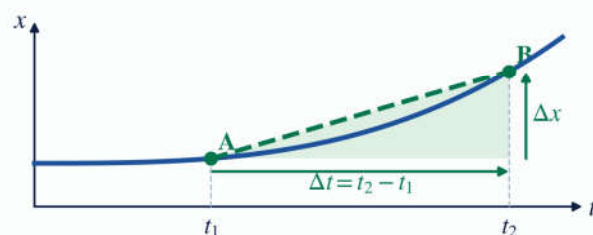
KEY IDEA

The sign specifies direction; the magnitude is the separation of the endpoints.

From an interval average to the velocity at an instant

The slope of $x(t)$ is built from measured position changes; direction distinguishes velocity from speed.

INTERVAL • AVERAGE VELOCITY

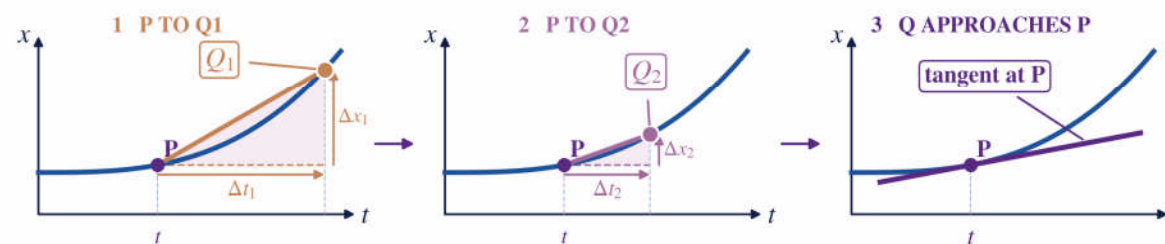


The signed slope of the secant AB

$$\bar{v}_x = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

Read the line AB:
rise $\div$ run, with sign

SHRINK THE INTERVAL • INSTANTANEOUS VELOCITY



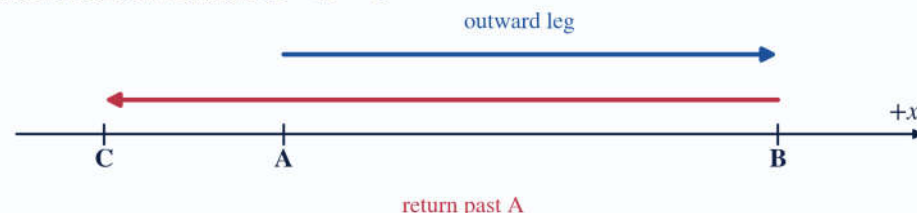
$Q_1 \rightarrow Q_2 \rightarrow P$: $\Delta t \rightarrow 0$ and the secant slope tends to the tangent slope.

$$v_x(t) = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$$

Limit of the secant slopes
= slope of the tangent at P

DIRECTION MATTERS • A TRIP WITH A TURNAROUND

Motion in one dimension: $A \rightarrow B \rightarrow C$



Final position C lies left of A: displacement is negative; total path length is positive.

VELOCITY • vector

$$\bar{v}_x = \frac{x_C - x_A}{\Delta t} < 0$$

Signed displacement
per elapsed time.

SPEED • scalar

$$\bar{u} = \frac{s_{AB} + s_{BC}}{\Delta t} > 0$$

Total distance traveled
per elapsed time.

Instantaneous relation: $u(t) = |\vec{v}(t)| \geq 0$; the two interval averages need not have equal magnitudes.

KEY IDEA

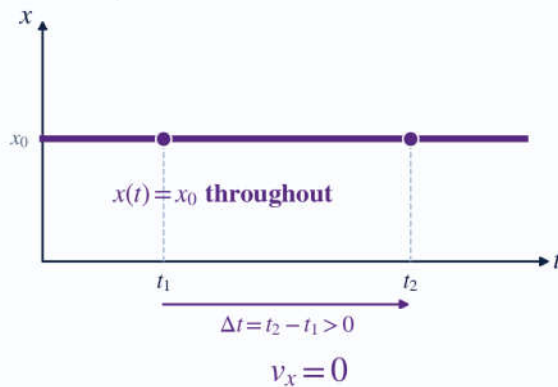
The secant measures an interval; the tangent is its limit; distance cannot cancel on a return trip.

Reading velocity from a position–time graph

For straight-line segments, the slope gives both the sign and the magnitude of velocity.

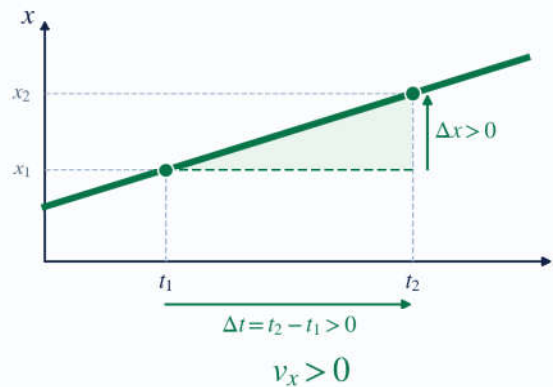
A • AT REST

Position stays constant.



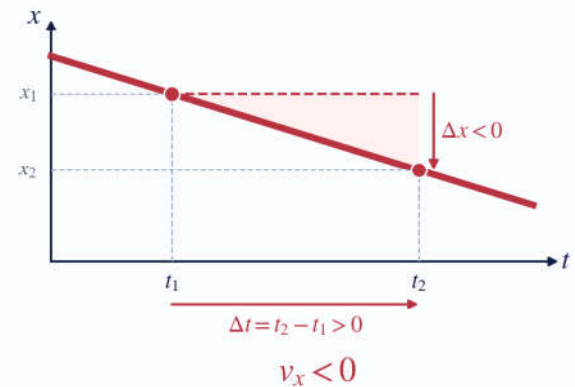
B • MOVING IN +x

Position increases with time.



C • MOVING IN -x

Position decreases with time.



THE MATHEMATICAL RULE

$$\bar{v}_x = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

For a straight line, every secant has the same slope.

$$x(t) = x_0 + v_x t, \quad v_x = \frac{dx}{dt} = \text{constant}$$

x_0 = initial position; units of slope: metres per second (m s^{-1}).

HOW TO INTERPRET THE SLOPE

Horizontal throughout → zero slope → at rest

Rising → positive slope → motion in +x

Falling → negative slope → motion in -x

Equal endpoints alone ($\Delta x = 0$) can also mean a return trip.

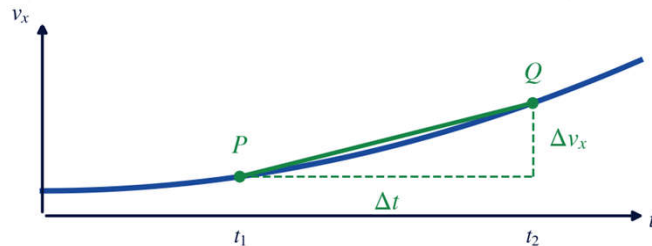
KEY IDEA

Position is read from the graph height; velocity from its slope. Always check the axes and units.

Average and instantaneous acceleration

Acceleration measures how velocity changes with time.

INTERVAL · AVERAGE ACCELERATION

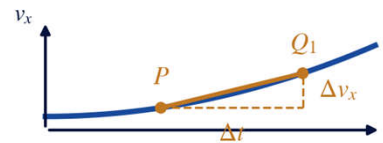


Secant slope between two times

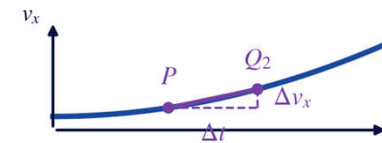
$$\bar{a}_x = \frac{v_x(t_2) - v_x(t_1)}{t_2 - t_1} = \frac{\Delta v_x}{\Delta t}$$

SHRINK THE INTERVAL · INSTANTANEOUS ACCELERATION

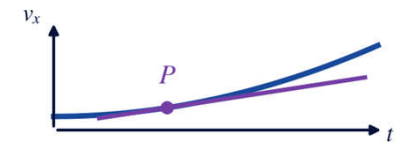
1 P TO Q1



2 P TO Q2



3 Q APPROACHES P



$\Delta t \rightarrow 0$

$$a_x(t) = \lim_{\Delta t \rightarrow 0} \frac{v_x(t + \Delta t) - v_x(t)}{\Delta t} = \frac{dv_x}{dt}$$

MATHEMATICAL DEFINITIONS · WHAT THE TWO SLOPES MEAN

AVERAGE · AN INTERVAL

$$\bar{a}_x = \frac{\Delta v_x}{\Delta t}$$

$$\Delta v_x = v_x(t_2) - v_x(t_1)$$

$$\Delta t = t_2 - t_1 > 0$$

Secant slope: velocity change per second.

INSTANTANEOUS · A TIME

$$a_x(t) = \frac{dv_x(t)}{dt}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{\Delta v_x}{\Delta t}$$

Tangent slope at the selected instant.

HOW TO READ THE GRAPH

$a_x > 0$ rising $v_x(t)$

$a_x < 0$ falling $v_x(t)$

$a_x = 0$ horizontal tangent

A steeper slope means a larger $|a_x|$.

KEY IDEA

The secant gives average acceleration; its limit is the tangent and acceleration at an instant.

Direction of velocity and acceleration in one dimension

The same direction increases speed. Opposite directions decrease speed.

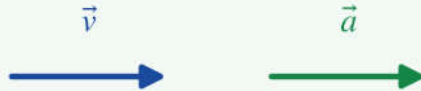
$$v_x \neq 0: \frac{d|v_x|}{dt} = \text{sgn}(v_x) a_x$$

$$\begin{aligned} \text{sgn}(v_x) &= +1 & \text{if } v_x > 0 \\ \text{sgn}(v_x) &= -1 & \text{if } v_x < 0 \end{aligned}$$

Same signs: speed increases.
Opposite signs: speed decreases.

ALIGNED VECTORS · SPEED INCREASES

$$v_x > 0, \quad a_x > 0$$



Rightward Motion: Speed increases.

$$v_x < 0, \quad a_x < 0$$



Leftward Motion: Speed increases.

OPPOSING VECTORS · SPEED DECREASES

$$v_x > 0, \quad a_x < 0$$



Rightward Motion: Speed decreases.

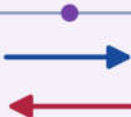
$$v_x < 0, \quad a_x > 0$$



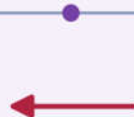
Leftward Motion: Speed decreases.

THE TURNING INSTANT

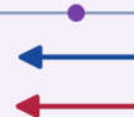
BEFORE
 $v_x > 0$



TURN
 $v_x = 0$



AFTER
 $v_x < 0$



$$v_x(t_*) = 0, \quad a_x(t_*) < 0$$

The velocity changes from rightward to leftward.

Acceleration still points left at the turn.

Speed decreases before and increases after the turn.

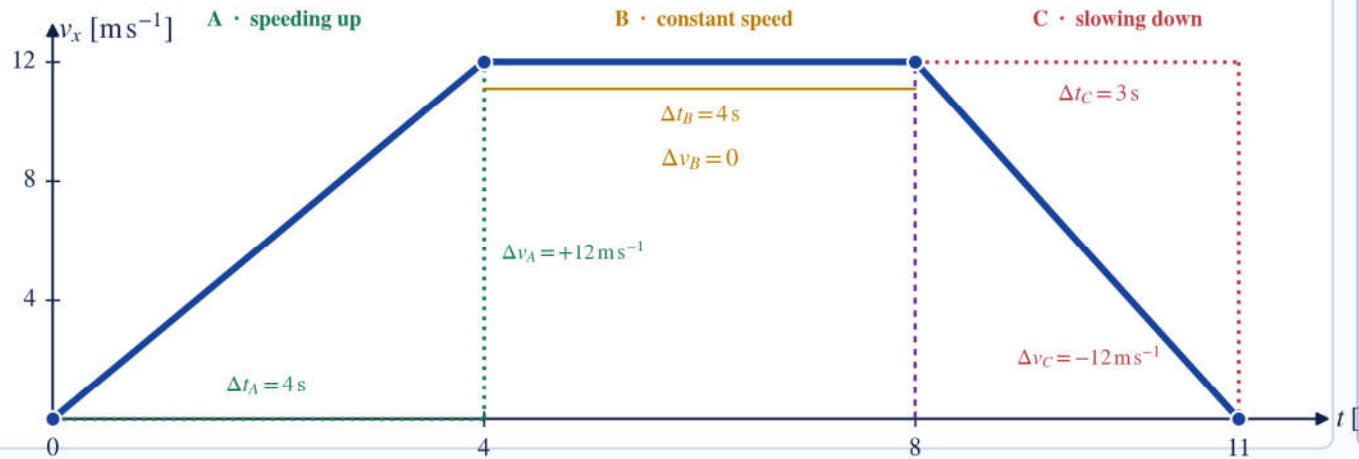
KEY IDEA

Acceleration changes velocity; the relative signs of velocity and acceleration determine the change in speed.

Read acceleration from a velocity–time graph

The slope measures the signed change in velocity per unit time. The height gives velocity.

ONE JOURNEY · THREE VELOCITY INTERVALS



THE DEFINITION

$$\bar{a}_x = \frac{\Delta v_x}{\Delta t} = \frac{v_x(t_2) - v_x(t_1)}{t_2 - t_1}$$

$$a_x(t) = \frac{dv_x}{dt}$$

Each straight segment has constant slope.

At a corner, the idealised graph has no unique instantaneous acceleration.

Unit of slope: m s^{-2} .

A · RISING VELOCITY

$$\bar{a}_A = \frac{12 - 0}{4 - 0} = +3 \text{ m s}^{-2}$$

$$v_x > 0, \quad a_x > 0$$

Moving right and gaining speed.

Velocity rises by 12 m/s in 4 s.

B · HORIZONTAL

$$\bar{a}_B = \frac{12 - 12}{8 - 4} = 0$$

$$v_x = 12 \text{ m s}^{-1} \neq 0$$

Moving right at constant speed.

Zero acceleration does not imply rest.

C · FALLING VELOCITY

$$\bar{a}_C = \frac{0 - 12}{11 - 8} = -4 \text{ m s}^{-2}$$

$$v_x > 0 \text{ for } 8 < t < 11 \text{ s}$$

Still moving right, but slowing down.

Velocity reaches zero at 11 s.

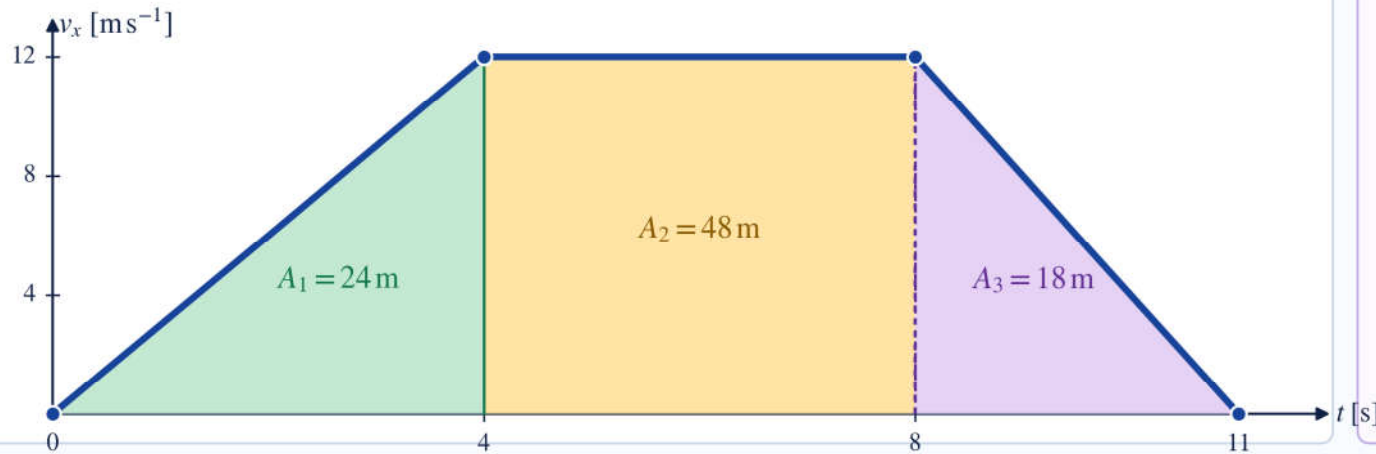
KEY IDEA

Slope is acceleration; graph height is velocity. Negative acceleration need not mean motion to the left.

Area under $v(t)$ gives displacement

The signed area adds each small contribution $v_x(t) dt$ to the change in position.

THE SAME JOURNEY • NOW READ THE AREA



ADD THE THREE AREAS

$$A_1 = \frac{1}{2}(4\text{ s})(12\text{ m s}^{-1}) = 24\text{ m}$$

$$A_2 = (4\text{ s})(12\text{ m s}^{-1}) = 48\text{ m}$$

$$A_3 = \frac{1}{2}(3\text{ s})(12\text{ m s}^{-1}) = 18\text{ m}$$

$$\Delta x = 24 + 48 + 18 = 90\text{ m}$$

DISPLACEMENT OR DISTANCE?

$$\Delta x = \int_0^{11} v_x(t) dt$$

$$s = \int_0^{11} |v_x(t)| dt$$

Here $v_x \geq 0$: both equal 90m.

Below the axis, signed area is negative.

THE MATHEMATICAL STATEMENT

$$\Delta x = x(t_2) - x(t_1)$$

$$= \int_{t_1}^{t_2} v_x(t) dt$$

$$dx = v_x(t) dt$$

Height $\times$ time has units of length:

$$(\text{m s}^{-1})\text{ s} = \text{m}.$$

READ THE SIGNS CORRECTLY

$a_x < 0$ on $8 < t < 11\text{ s}$:

the graph falls, so velocity decreases.

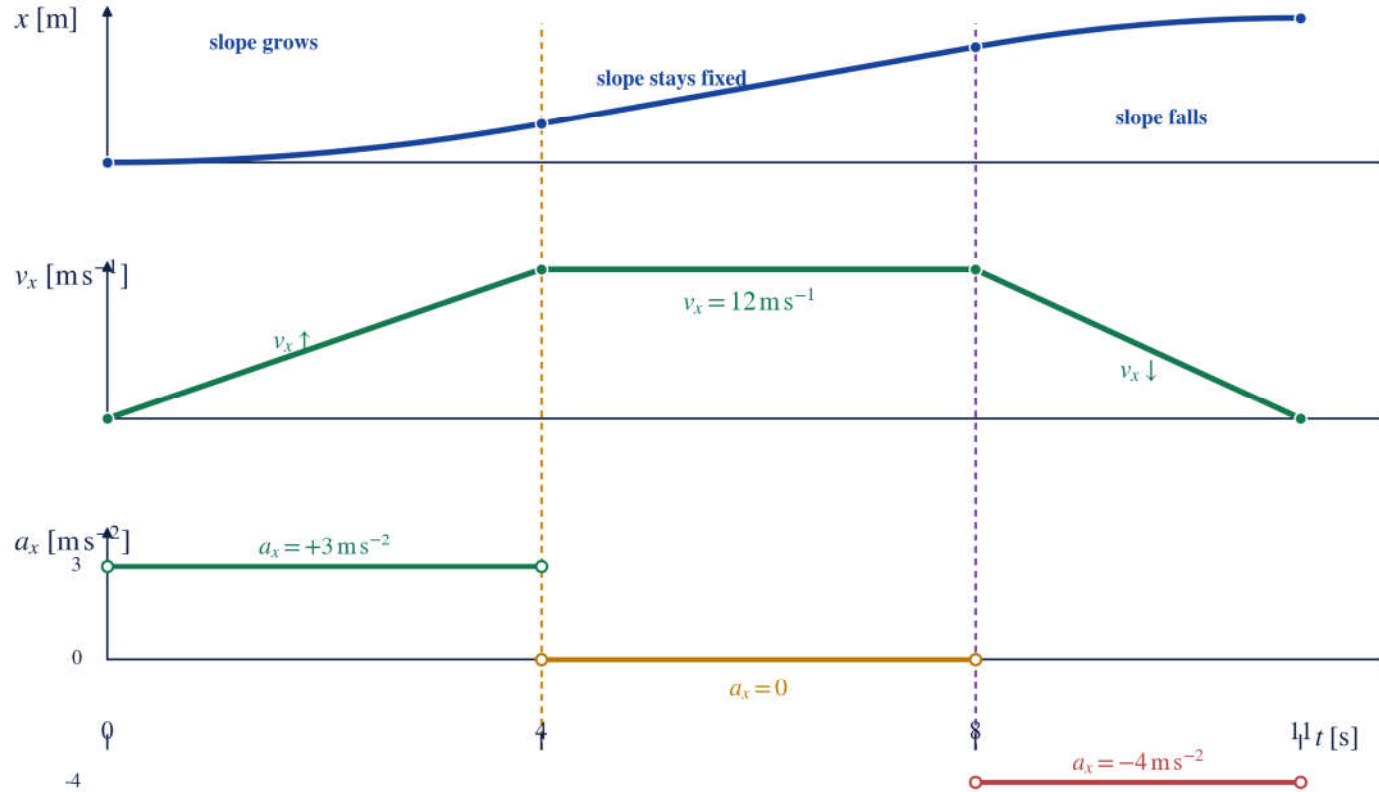
$v_x > 0$ until $t = 11\text{ s}$:

area stays positive despite falling v_x .

One motion, three synchronized graphs

Read the same journey as position, velocity, and acceleration; their slopes connect the three graphs.

ONE JOURNEY · THREE GRAPHS ON THE SAME CLOCK



DERIVATIVES · READ THE SLOPE

$$v_x(t) = \frac{dx}{dt}$$

$$a_x(t) = \frac{dv_x}{dt}$$

Slope of $x(t)$ gives velocity.

INTEGRALS · ADD THE CHANGES

$$\Delta v_x = \int_{t_1}^{t_2} a_x(t) dt$$

$$\Delta x = \int_{t_1}^{t_2} v_x(t) dt$$

The signed areas restore the change.

FOLLOW THE TIME INTERVALS

$0 < t < 4$: $a_x = +3$; v_x rises.

$4 < t < 8$: $a_x = 0$; $v_x = 12$.

$8 < t < 11$: $a_x = -4$; v_x falls.

At 4 s and 8 s, acceleration is undefined in this idealised model.

KEY IDEA

The derivative moves downward through the stack; the signed integral moves upward.

Four equations for motion at constant acceleration

One-dimensional motion with constant a_x : choose the equation that omits the quantity you do not know.

$$t_0 = 0, \quad x_0 = x(0), \quad v_{x0} = v_x(0), \quad \Delta x = x - x_0, \quad a_x = \text{constant}$$

All quantities refer to the same +x axis.

1 VELOCITY FROM a_x AND TIME

$$v_x = v_{x0} + a_x t$$

Constant a_x gives a linear rise or fall of v_x .

DERIVATION

$$\int_{v_{x0}}^{v_x} dv_x = \int_0^t a_x d\tau$$

$$v_x - v_{x0} = a_x t$$

WHEN TO USE IT

Known v_{x0}, a_x, t

Find v_x

Position not needed

2 POSITION FROM a_x AND TIME

$$x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2$$

Initial motion plus the displacement due to a_x .

DERIVATION

$$\int_{x_0}^x dx = \int_0^t (v_{x0} + a_x \tau) d\tau$$

$$x - x_0 = v_{x0}t + \frac{1}{2}a_x t^2$$

WHEN TO USE IT

Known x_0, v_{x0}

and a_x, t

Find x or Δx

3 VELOCITY WITHOUT TIME

$$v_x^2 = v_{x0}^2 + 2a_x(x - x_0)$$

Connects a_x and Δx ; the square loses sign.

DERIVATION

$$a_x t = v_x - v_{x0}, \quad 2\Delta x = (v_{x0} + v_x)t$$

$$2a_x \Delta x = (v_{x0} + v_x)(v_x - v_{x0})$$

WHEN TO USE IT

Known v_{x0}, a_x

and Δx

Choose sign separately

4 DISPLACEMENT FROM MEAN VELOCITY

$$x - x_0 = \frac{v_{x0} + v_x}{2} t$$

For linear $v_x(t)$, mean velocity $\times$ time is Δx .

DERIVATION

$$\Delta x = \int_0^t v_x(\tau) d\tau$$

$$\bar{v}_x = \frac{v_{x0} + v_x}{2}, \quad \Delta x = \bar{v}_x t$$

WHEN TO USE IT

Known v_{x0}, v_x, t

Find Δx

a_x not needed

KEY IDEA

Choose from the known variables. These relations require constant acceleration over the entire chosen interval.

A Reliable Kinematics Workflow

A correct calculation begins with a physical representation and ends with an independent check—not with immediate substitution.

SOLVE EVERY PROBLEM IN SIX PASSES

1 REPRESENT THE MOTION

Sketch the object, important events, and the time interval.

2 DEFINE THE REFERENCE FRAME

Choose origin, + direction, clock zero, and state the observer.

3 TRANSLATE WORDS INTO VARIABLES

Write x_0 , x , Δx , v_0 , v , a , and t with signs and units.

4 CHOOSE THE PHYSICAL MODEL

Decide whether a is constant; split the motion if the rule changes.

5 SELECT AND SOLVE SYMBOLICALLY

Use an equation containing the target and known quantities only.

6 SUBSTITUTE, VERIFY, INTERPRET

Insert numbers last; check units, sign, magnitude, and meaning.

BEFORE WRITING AN EQUATION

EVENT MAP

start $\rightarrow$ change $\rightarrow$ finish
What defines each event?

SIGN MAP

Which way is +?
Do v and a agree?

MODEL TEST

Is a constant here?
One interval or several?

DURING THE CALCULATION — KEEP THE PHYSICS VISIBLE

Solve algebraically before inserting numbers: the structure remains visible and units are easier to track. Carry signed quantities; do not replace v or a by magnitudes unless the question explicitly asks for speed. Attach SI units to every substituted value. Dimensional inconsistency exposes an incorrect equation or algebraic step. Retain guard digits during calculation; round only the final reported result to justified significant figures.

AFTER THE CALCULATION — FOUR INDEPENDENT CHECKS

UNITS

Does the result have the required dimension?

SIGN

Does its direction match the chosen coordinate?

LIMIT

Does $a \rightarrow 0$ recover constant-velocity motion?

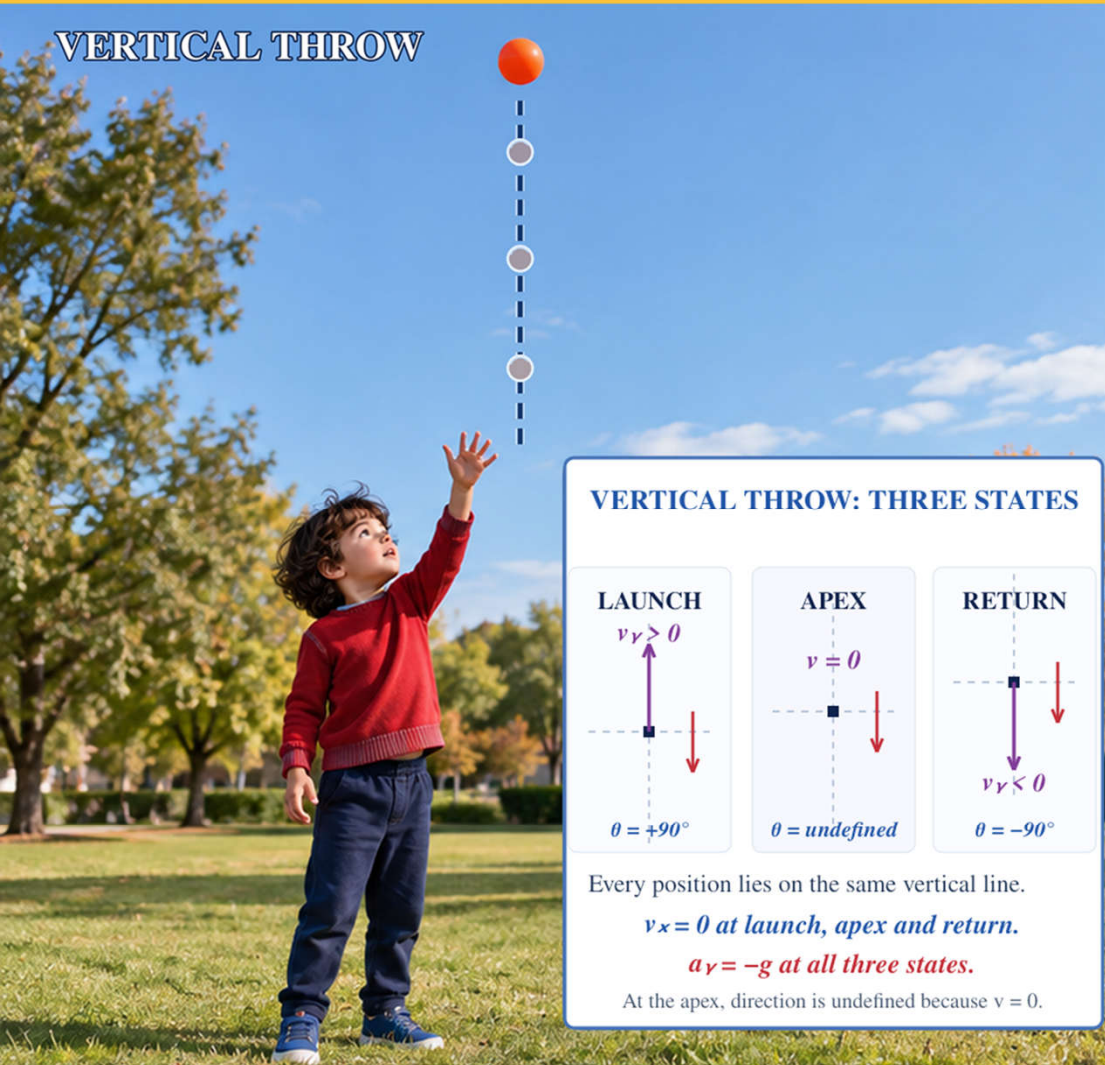
SCALE

Is the magnitude physically plausible?

If a check fails, return to the representation or model—not merely to the calculator.

Projectile Motion in Practice

VERTICAL THROW



A ball rises and returns along one vertical line.

BASKETBALL SHOT

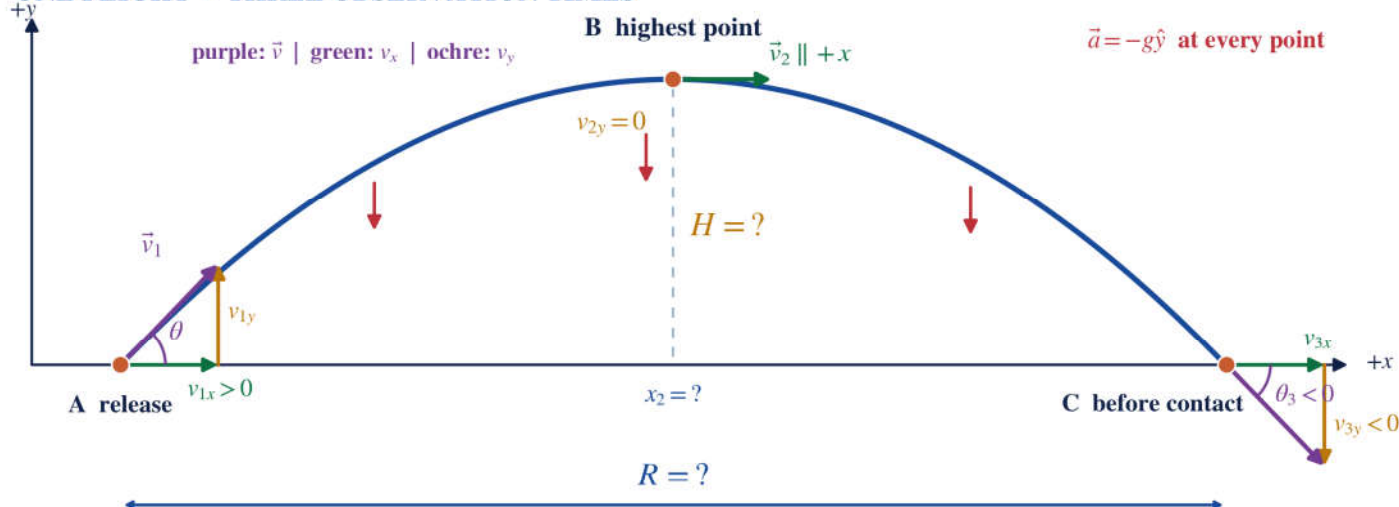


The ball moves horizontally as it rises and falls.

Projectile motion: three states and the questions to solve

Known: v_0 , launch angle θ , and $g > 0$. Choose $+x$ forward and $+y$ upward. Neglect air resistance.

ONE FLIGHT · THREE OBSERVATION TIMES



A RELEASE

$$t_1 = 0, \quad (x_1, y_1) = (0, 0)$$

$$v_{1x} = v_0 \cos \theta > 0$$

$$v_{1y} = v_0 \sin \theta > 0$$

$$\vec{v}_1 = v_{1x}\hat{x} + v_{1y}\hat{y}$$

$$a_x = 0, \quad a_y = -g$$

Released upward and forward

B HIGHEST POINT

$$t_2 = ?, \quad (x_2, y_2) = (?, H)$$

$$v_{2x} = v_{1x} > 0$$

$$v_{2y} = 0$$

$$\vec{v}_2 = v_{2x}\hat{x}$$

$$a_x = 0, \quad a_y = -g$$

Moves right; still accelerates down

C BEFORE CONTACT

$$t_3 = ?, \quad (x_3, y_3) = (R, 0)$$

$$v_{3x} = v_{1x} > 0$$

$$v_{3y} < 0, \quad |v_{3y}| = ?$$

$$\vec{v}_3 = v_{3x}\hat{x} + v_{3y}\hat{y}$$

$$a_x = 0, \quad a_y = -g$$

Velocity points right and down

WHAT MUST WE DETERMINE?

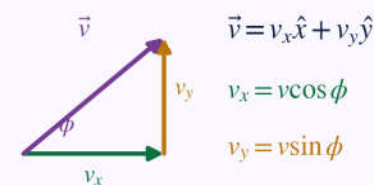
At the highest point B

- 01 t_2 : time to the top
- 02 $H = y_2$: maximum height
- 03 x_2 : horizontal position
- 04 $\vec{a}_2$: acceleration there

Just before contact C

- 05 t_3 : total flight time
- 06 $R = x_3$: horizontal range
- 07 $\vec{v}_3$: impact velocity

GENERAL VECTOR DECOMPOSITION



At A: $\phi = \theta > 0$, so $v_y > 0$.

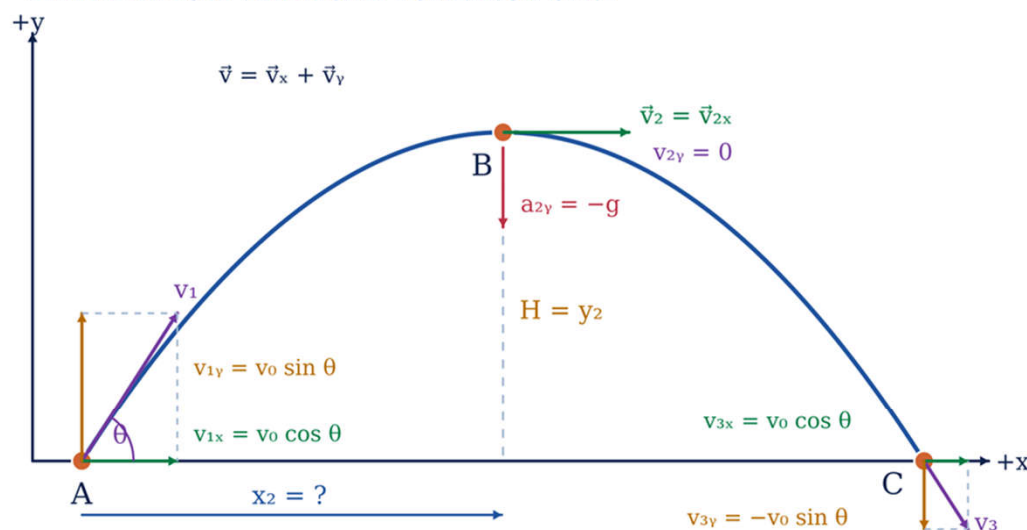
At B: $v_y = 0$; the ball still moves right.

At C: $\phi < 0$, so $v_y < 0$.

Solution I — time, height, and position at the apex

Same launch and landing height; $v_0 > 0$, $0 < \theta < 90^\circ$, $g > 0$; air resistance neglected.

THE APEX IS A VERTICAL TURNING POINT



DERIVE THE THREE UNKNOWN QUANTITIES

Vertical velocity: $v_y(t) = v_0 \sin \theta - gt$

At the apex, $v_y(t_2) = 0$:

$$0 = v_0 \sin \theta - gt_2 \implies t_2 = \frac{v_0 \sin \theta}{g}$$

Vertical position: $y(t) = v_0 \sin \theta t - \frac{1}{2}gt^2$

$$H = y(t_2) = \frac{v_0^2 \sin^2 \theta}{2g}$$

Horizontal position: $x(t) = v_0 \cos \theta t$

$$x_2 = v_0 \cos \theta t_2 = \frac{v_0^2 \sin \theta \cos \theta}{g}$$

01 WHY DOES v_y VANISH?

$a_y = -g$ means v_y drops by g each second.

Initial value: $v_{1y} = v_0 \sin \theta > 0$.

At the turning point: $v_{2y} = 0$.

Horizontal motion continues: $v_{2x} > 0$.

02 ACCELERATION AT B

$$a_x = 0, \quad a_y = -g$$

$$\vec{a}_2 = (0, -g)$$

Gravity acts downward at every point.

$v_{2y} = 0$ does not imply $a_{2y} = 0$.

03 WHAT WE LEARN

$t_2 = v_{1y}/g$: vertical speed sets the time.

$$H = v_{1y}t_2 - \frac{1}{2}gt_2^2.$$

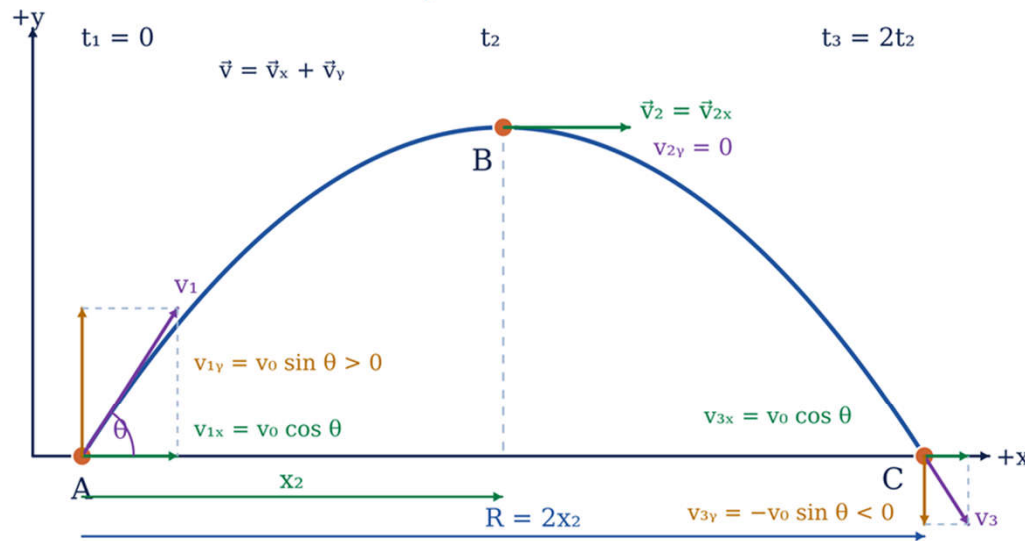
$H = v_{1y}^2/(2g)$: substitute t_2 .

$x_2 = v_{1x}t_2$: horizontal speed times time.

Solution II — flight time and horizontal range

Landing at the launch height: $y_3 = y_1 = 0$. The nonzero root of $y(t) = 0$ gives the impact time.

TWO HEIGHT-ZERO EVENTS; ONE FLIGHT



SOLVE THE LANDING CONDITION

$$0 = y(t_3) = v_0 \sin \theta t_3 - \frac{1}{2} g t_3^2$$

$$0 = t_3 \left(v_0 \sin \theta - \frac{1}{2} g t_3 \right)$$

$t_3 = 0$ is release; the second root is landing:

$$t_3 = \frac{2v_0 \sin \theta}{g} = 2t_2$$

Horizontal motion is uniform: $x(t) = v_0 \cos \theta t$.

$$R = x(t_3) = v_0 \cos \theta \frac{2v_0 \sin \theta}{g}$$

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

04 THE TWO ROOTS

$$y(t) = t(v_{1y} - gt/2) = 0.$$

First root $t = 0$: release from $y = 0$.

$$\text{Second root } v_{1y} - gt_3/2 = 0.$$

The second root marks landing at $y = 0$.

05 WHY $R = 2x_2$?

$t_3 = 2t_2$ follows from the two roots.

Constant $v_x = v_0 \cos \theta$.

$$x_2 = v_x t_2, \text{ while } R = v_x t_3.$$

Hence $R = 2x_2$ only for equal heights.

06 WHEN IS THIS VALID?

Launch and landing levels must be equal.

Air drag is ignored: $a_x = 0$.

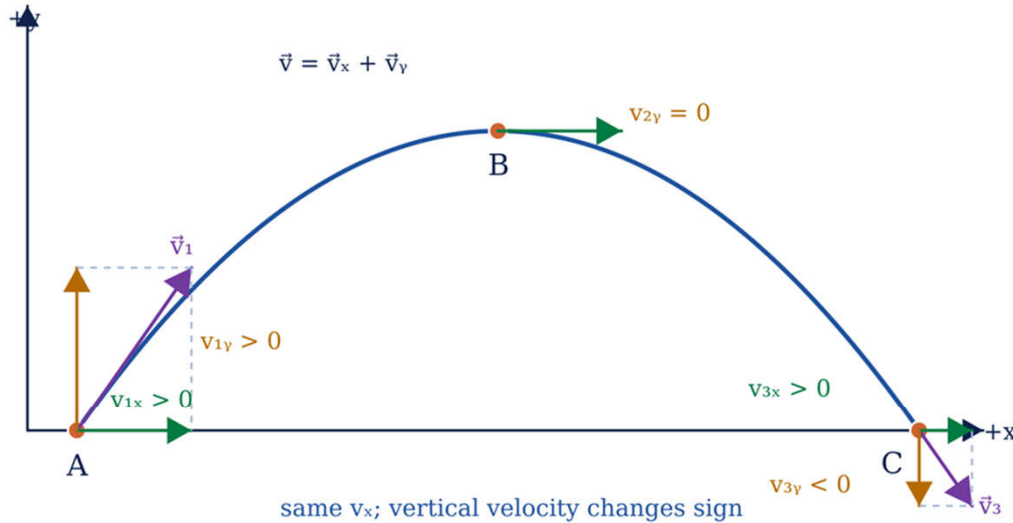
$a_y = -g$ is constant over the flight.

Unequal heights: solve $y(t_3) = y_3$ anew.

Solution III — impact velocity and physical interpretation

The horizontal component stays constant; the vertical component reverses sign at equal launch and landing heights.

COMPARE THE VELOCITY VECTORS



COMPUTE THE VELOCITY AT CONTACT

$$v_{3x} = v_0 \cos \theta \text{ (horizontal acceleration is zero)}$$

$$\begin{aligned} v_{3y} &= v_0 \sin \theta - gt_3 \\ &= v_0 \sin \theta - g \frac{2v_0 \sin \theta}{g} = -v_0 \sin \theta \end{aligned}$$

$$\vec{v}_3 = \vec{v}_{3x} + \vec{v}_{3y} = (v_0 \cos \theta, -v_0 \sin \theta)$$

$$|\vec{v}_3| = \sqrt{v_{3x}^2 + v_{3y}^2} = v_0$$

$$\phi_3 = -\theta \text{ (measured from the positive } x \text{ axis)}$$

$$\vec{a}_3 = (0, -g) \text{ even at contact.}$$

07 WHAT IS THE DIRECTION?

$v_{3x} > 0$: forward motion continues.

$v_{3y} < 0$: motion is downward.

$$\tan \phi_3 = v_{3y}/v_{3x} = -\tan \theta.$$

$\phi_3 = -\theta$: downward launch mirror.

SPEED VERSUS VELOCITY

$$|\vec{v}_3|^2 = v_{3x}^2 + v_{3y}^2.$$

$$= v_0^2 (\cos^2 \theta + \sin^2 \theta).$$

$$|\vec{v}_3| = v_0: \text{ same speed at both heights.}$$

But $\vec{v}_1 \neq \vec{v}_3$: direction differs.

CONSISTENCY CHECK

$t_3 = 2t_2$: the vertical path is symmetric.

$R = 2x_2$: horizontal speed is constant.

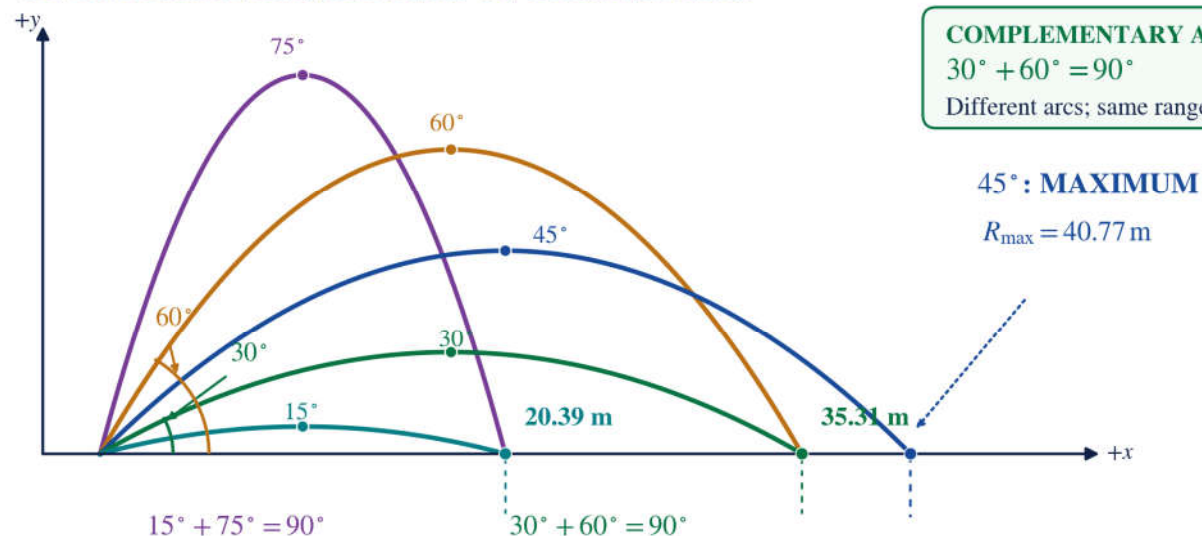
$v_{3y} = -v_{1y}$ at equal heights.

With air drag these equalities need not hold.

Five launch angles — paired ranges and a unique maximum

One launch speed: $v_0 = 20 \text{ m/s}$. Compare $15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ$; level ground and $g = 9.81 \text{ m/s}^2$.

FIVE PHYSICAL TRAJECTORIES ON THE SAME AXES



WHY DO THE PAIRS LAND TOGETHER?

$$T = \frac{2v_0 \sin \theta}{g}, \quad v_x = v_0 \cos \theta$$

$$R = v_x T = \frac{v_0^2}{g} 2 \sin \theta \cos \theta$$

$$= \frac{v_0^2}{g} \sin(2\theta)$$

If $\theta_2 = 90^\circ - \theta_1$:

$$\sin(2\theta_2)$$

$$= \sin(180^\circ - 2\theta_1)$$

$$= \sin(2\theta_1) \Rightarrow R_2 = R_1$$

Peak: $\sin(2\theta) \leq 1 \Rightarrow \theta = 45^\circ$.

01 TWO COMPLEMENTARY PAIRS

$$R_{15} = R_{75} = 20.39 \text{ m.}$$

$$R_{30} = R_{60} = 35.31 \text{ m.}$$

$$R_{45} = 40.77 \text{ m is largest.}$$

Each landing point is marked on the axis.

02 WHY THE FLIGHTS DIFFER

15° : large v_x , small v_y ; low arc.

75° : small v_x , large v_y ; high arc.

$R = v_x T$: the factors compensate exactly.

Flight times and apex heights are unequal.

03 THE SINGLE OPTIMAL ANGLE

45° maps to itself: $90^\circ - 45^\circ = 45^\circ$.

$v_x = v_y$ at 45° , so $2v_x v_y$ peaks.

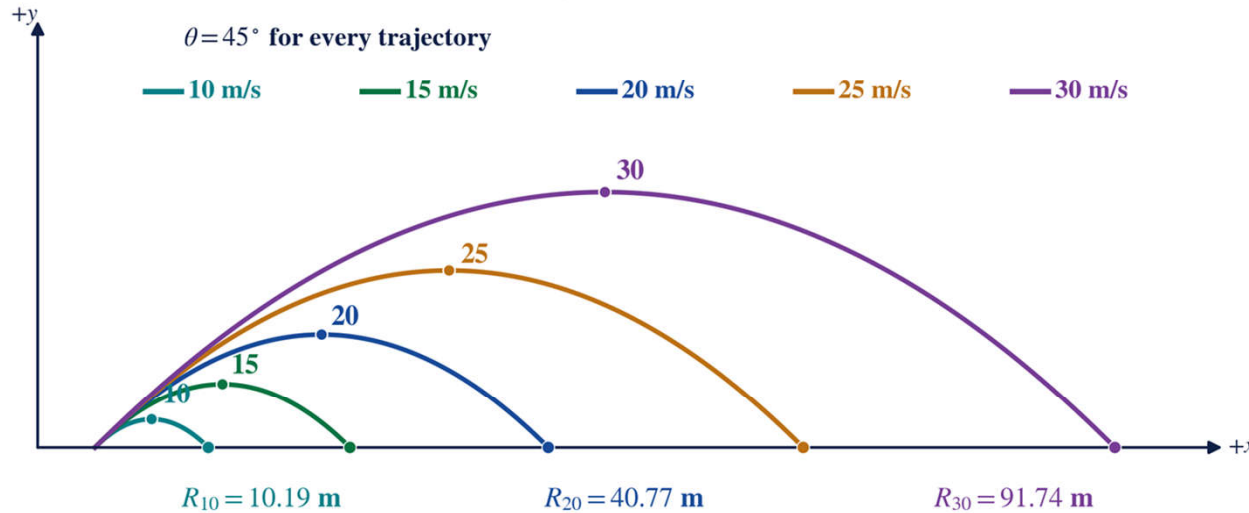
Maximum range holds for level ground.

Air drag or unequal heights change the result.

Five launch speeds — one angle and a scaling law

One launch angle: $\theta = 45^\circ$. Compare $v_0 = 10, 15, 20, 25, 30$ m/s; level ground, no drag, $g = 9.81$ m/s².

FIVE TRAJECTORIES; COMMON x AND y SCALE



WHY DOES THE PATH EXPAND?

At fixed θ , let $v_B = kv_A$:

$$T = \frac{2v_0 \sin \theta}{g} \Rightarrow T_B = kT_A$$

$$R = v_x T; v_{x,B} = kv_{x,A}$$

$$R_B = k^2 R_A$$

$$H = \frac{v_{0y}^2}{2g} \Rightarrow H_B = k^2 H_A$$

$$30/10 = 3: T \times 3, R \times 9, H \times 9.$$

Same direction; greater magnitude at launch.

01 FLIGHT TIME

$$T = 2v_0 \sin 45^\circ / g.$$

For 10, 15, 20, 25, 30 m/s:

$$T = 1.44, 2.16, 2.88, 3.60, 4.32 \text{ s.}$$

Time grows in direct proportion to v_0 .

02 RANGE

$$R = v_0^2 / g \text{ at } 45^\circ.$$

$$R = 10.19, 22.94, 40.77,$$

$$63.71, 91.74 \text{ m.}$$

Doubling v_0 makes the range fourfold.

03 MAXIMUM HEIGHT

$$H = v_0^2 / (4g) \text{ at } 45^\circ.$$

$$H = 2.55, 5.73, 10.19,$$

$$15.93, 22.94 \text{ m.}$$

At fixed angle, height also scales as v_0^2 .

Worked example 4 • Emergency braking of an autonomous shuttle

A sensor delay and limited floor traction turn a routine stop into an engineering safety decision.

ONE VEHICLE • FOUR MOTION PHASES • ONE OBSTACLE

+x forward

Obstacle at $x = 118 \text{ m}$

Sensor observes the obstacle

The shuttle starts from rest at $x = 0$.



1 • ACCELERATE

0–6 s

2 • CRUISE

6–10 s

3 • SENSOR DELAY

10–10.5 s

4 • BRAKE

until rest

MOTION COMMAND

$$v_0 = 0, \quad x_0 = 0$$

$$a_1 = +2 \text{ m s}^{-2}$$

$$\Delta t_1 = 6 \text{ s}$$

$$\Delta t_2 = 4 \text{ s at constant } v$$

One common clock runs through every phase.

SENSOR AND FLOOR

$$\tau = 0.5 \text{ s without braking}$$

$$a_{\text{dry}} = -3 \text{ m s}^{-2}$$

$$a_{\text{wet}} = -2 \text{ m s}^{-2}$$

Braking acceleration is taken as constant.

Neglect vehicle length and air drag.

ENGINEERING QUESTIONS

Where does braking begin?

Where does the shuttle stop?

Is there a dry-floor safety margin?

What changes on a wet floor?

Find the minimum required deceleration.

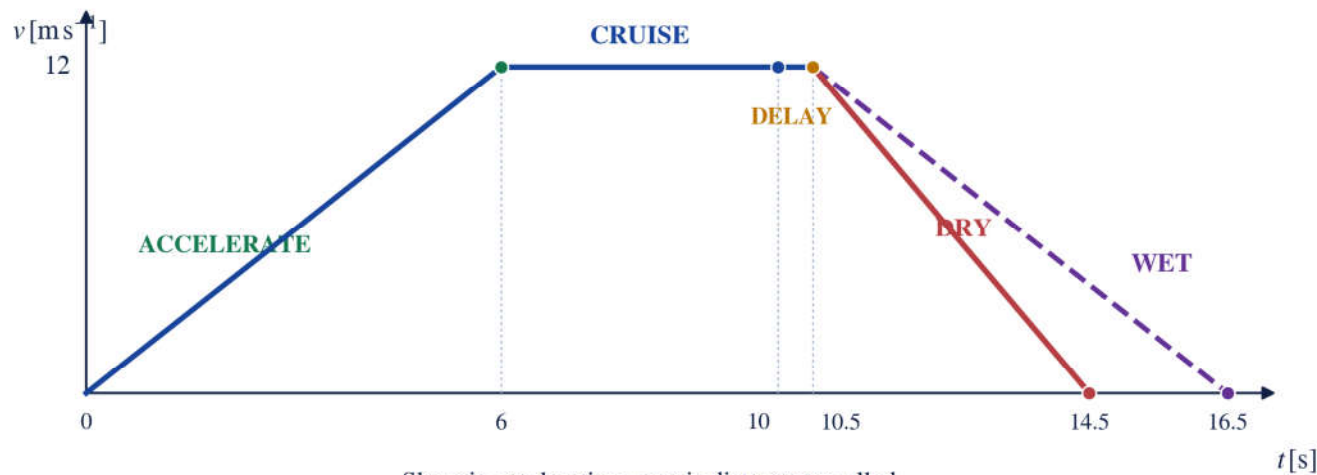
KEY IDEA

State the sign convention, propagate each endpoint, then compare stopping position with the obstacle.

Worked example 4 • Solve and verify the emergency stop

The same kinematic equations are applied separately on each constant-acceleration interval.

VELOCITY–TIME HISTORY • DRY AND WET BRAKING



1 • ENDPOINTS BEFORE BRAKING

$$v_6 = 0 + (2)(6) = 12 \text{ m s}^{-1}$$

$$x_6 = \frac{1}{2}(2)(6^2) = 36 \text{ m}$$

$$x_{10} = 36 + (12)(4) = 84 \text{ m}$$

$$x_b = 84 + (12)(0.5) = 90 \text{ m}$$

2 • BRAKING FROM 12 m/s

$$t_{b, \text{dry}} = \frac{0 - 12}{-3} = 4 \text{ s}$$

$$d_{b, \text{dry}} = \frac{0^2 - 12^2}{2(-3)} = 24 \text{ m}$$

$$x_{\text{stop, dry}} = 90 + 24 = 114 \text{ m}$$

$$d_{b, \text{wet}} = 36 \text{ m} \Rightarrow x_{\text{free-stop}} = 126 \text{ m}$$

THE FOUR EQUATIONS IN USE

$$v = v_0 + at$$

$$\Delta x = v_0 t + \frac{1}{2}at^2$$

$$v^2 = v_0^2 + 2a\Delta x$$

$$\Delta x = \frac{1}{2}(v_0 + v)t$$

SAFETY DECISION

$$\text{Obstacle } x_O = 118 \text{ m}$$

$$\text{Dry margin} = 118 - 114 = 4 \text{ m}$$

$$\text{Wet free-stop} = 126 \text{ m}$$

$$v_{\text{hit}} = \sqrt{12^2 - 2(2)(28)} = 5.66 \text{ m/s}$$

CHECK AND DESIGN LIMIT

$$d_{b, \text{dry}} = \frac{12+0}{2}(4) = 24 \text{ m}$$

$$d_{\text{available}} = 118 - 90 = 28 \text{ m}$$

$$|a|_{\text{min}} = \frac{12^2}{2(28)} = \frac{18}{7} \approx 2.57 \text{ m s}^{-2}$$

A 4 m margin requires 3 m/s².

KEY IDEA

The 126 m wet-floor stopping position is a no-obstacle prediction; the actual vehicle meets the barrier first.

Correct Algebra Can Still Describe the Wrong Motion

A physically reliable solution must pass independent checks on the frame, event, model, units and limiting behaviour.

RUN A FIVE-TEST AUDIT BEFORE ACCEPTING THE ANSWER

Do not ask only “Did I manipulate the equation correctly?” Ask whether the symbols still describe the stated motion. Independent checks catch different failure modes; agreement between them is stronger evidence than repeating the same calculation.



FRAME

Origin, + direction
and clock defined?



EVENT

What condition
defines the answer?



MODEL

Is a constant?
One interval?



UNITS

Does every term
share dimensions?



LIMIT

Does $a \rightarrow 0$ or
 $t \rightarrow 0$ make sense?

Accept the result only when the calculation and the physical story agree.

ERROR 1 — $v = 0$ DOES NOT MEAN $a = 0$

Wrong at the top: $v_y = 0 \Rightarrow a_y = 0$

Repair: with +y upward, $a_y = -g$ throughout.

Velocity describes the instantaneous motion.
Acceleration describes how velocity is changing.
They are different physical quantities.

ERROR 2 — WRONG EVENT CONDITION

Wrong catch condition: $v_1(t) = v_2(t)$

Repair: meeting means $x_1(t) = x_2(t)$.

For ground impact use $y(t) = y_{\text{ground}}$.
For the highest point use $v_y(t) = 0$.
Translate the event before selecting equations.

WHY USE INDEPENDENT CHECKS?

Substituting an answer back into the same equation checks arithmetic, but not necessarily the model or event definition. A second physical representation tests different assumptions.

STOP AND REBUILD THE MODEL IF...

- the acceleration changes within the chosen interval
- the coordinate direction changes without updating signs
- the requested event is not written as an equation
- a vector equation is replaced by unsigned magnitudes
- the answer violates units, scale or an obvious limiting case

ERROR 3 — FORMULA OUTSIDE ITS MODEL

Wrong: use $R = v_0^2 \sin(2\theta)/g$ for unequal heights.

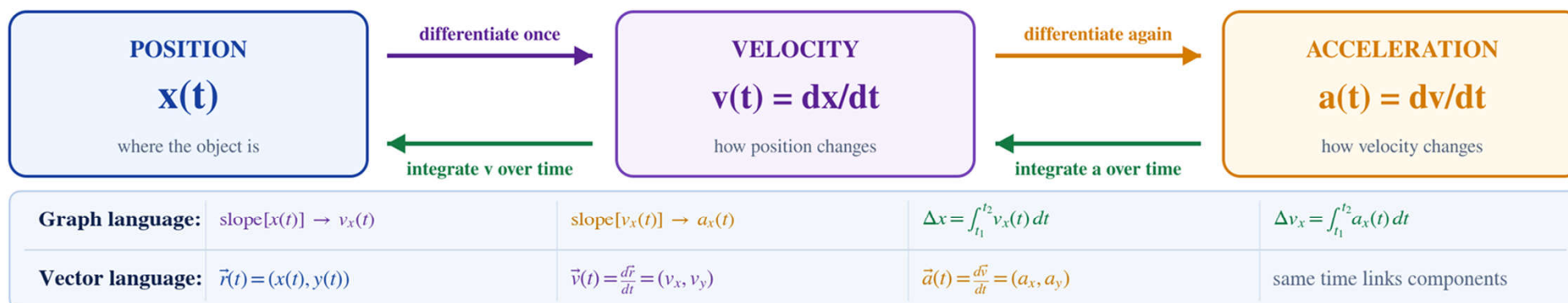
Repair: solve $y(t) = y_{\text{landing}}$, then $x(t)$.

State assumptions: constant acceleration, one frame, negligible drag and the relevant time interval.
A formula is compressed model information—not a rule without conditions.

Kinematics Is a Connected System of Representations

A solution is trustworthy when words, diagrams, graphs, equations, units and signs describe the same motion.

THE CENTRAL MATHEMATICAL CHAIN — READ IT IN BOTH DIRECTIONS



1 DEFINE THE FRAME

Choose the observer and origin.
 Define positive directions.
 Set the zero of time.
 Attach signs to components.
 A coordinate is not a distance.

2 DEFINE THE EVENT

Translate the requested event:
 meeting $\rightarrow x_1(t) = x_2(t)$
 highest point $\rightarrow v_y(t) = 0$
 impact $\rightarrow y(t) = y_{\text{ground}}$
 Event equations select the time.

3 CHOOSE THE MODEL

Decide what is constant.
 Split motion when rules change.
 Use components in 2D.
 State assumptions explicitly.
 A formula inherits all assumptions.

4 SOLVE AND VERIFY

Solve symbolically first.
 Carry units and signed values.
 Reject nonphysical roots.
 Check dimensions, limits and scale.
 Interpret direction and meaning.