

Measurement Methods in Physics

Lecture 5

Uncertainty in Direct Measurements II: Systematic Errors and Reporting Measurement Results



Where are we?



Previous lectures

- Measurement resolution
- Random fluctuations
- Repeated measurements
- Mean and standard deviation
- Uncertainty of the mean

Today

- Systematic errors
- Consistency of measurements
- Weighted mean
- Standard uncertainty
- Reporting measurement results
- Relative uncertainty
- Repeatability, reproducibility and accuracy

Recap: random uncertainty

From repeated measurements

■ Sample mean

$$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

■ Sample standard deviation

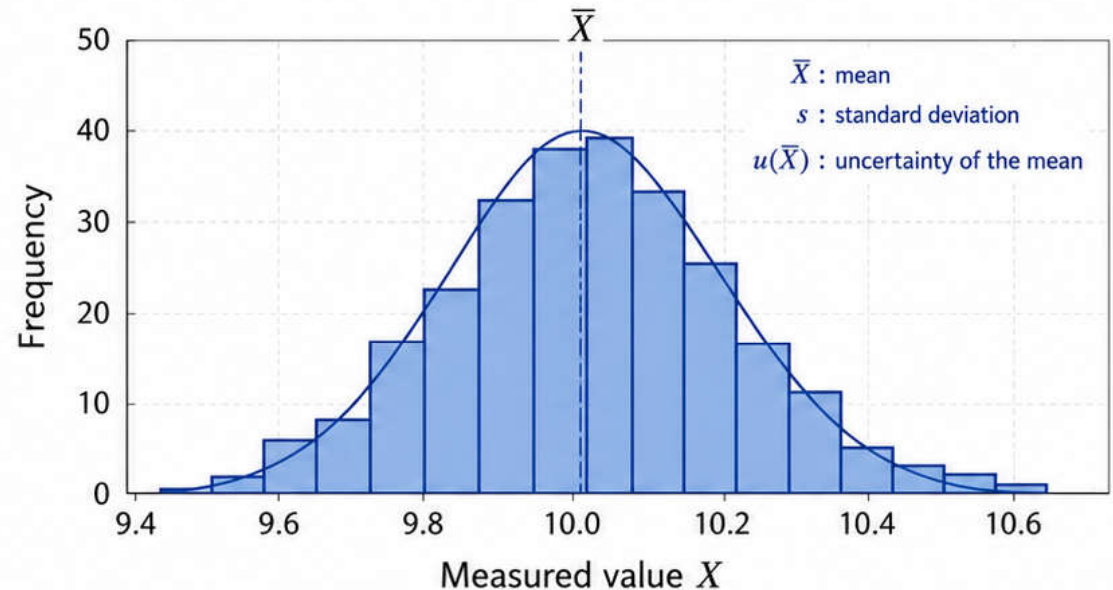
$$s = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (X_i - \bar{X})^2}$$

■ Standard uncertainty of the mean

$$u(\bar{X}) = \frac{s}{\sqrt{N}}$$

- Repetition allows us to estimate the random component of uncertainty statistically.
- The dispersion of the results is characterized by s .
- The uncertainty of the mean decreases as $1/\sqrt{N}$.

Example: distribution of repeated measurements



Key message:

Increasing the number of measurements (N) reduces random uncertainty – but **not systematic error**.

Learning objectives

By the end of this lecture, you should be able to:



1. distinguish random and systematic effects;



2. identify possible sources of systematic error;



3. test the consistency of measurements;



4. combine measurements using a weighted mean;



5. distinguish Type A and Type B evaluation;



6. calculate absolute and relative uncertainty;



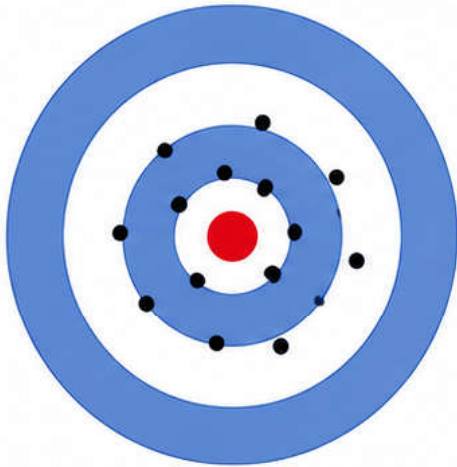
7. correctly report a measurement result;



8. distinguish repeatability, reproducibility and accuracy.

Random vs. systematic errors

Random effects



- Vary from measurement to measurement
- Produce dispersion (large scatter)
- Cause uncertainty (random noise)
- Evaluated statistically from data
- Reduced by increasing the number of measurements

Affect precision (repeatability)

Systematic effects



- Shift measurements in the same direction
- Produce a bias (not random scatter)
- Cause error (systematic bias)
- Not revealed by repetition under identical conditions
- Not reduced by increasing the number of measurements

Affect accuracy (closeness to the true value)



Key message:

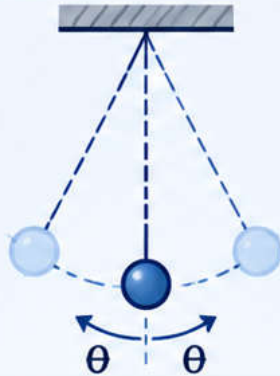
Random effects produce scatter around the mean; systematic effects shift all results.

Systematic errors: 1. Example – pendulum

1 We measure g with a pendulum

For small oscillations, the period is (approximately)

$$T = 2\pi\sqrt{\frac{l}{g}}$$



$$g = \frac{4\pi^2 l}{T^2}$$

2 What assumptions are hidden in this equation?

- small amplitudes ($\theta \ll 1$),
- point mass (no moment of inertia),
- rigid and massless string,
- no air resistance,
- constant gravitational field.



3 A more accurate model

Taking into account finite amplitude and the moment of inertia:

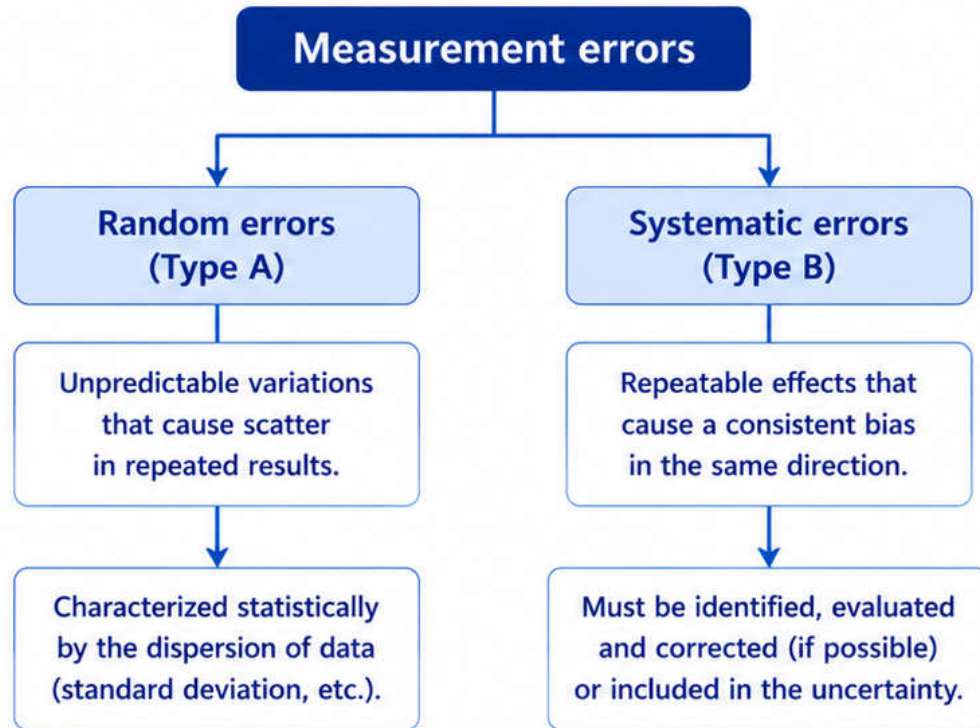
$$T \approx 2\pi\sqrt{\frac{l}{g} \cdot \left(1 + \frac{\theta^2}{16} + \dots\right)}$$

? What happens if these assumptions are not fully valid?



An oversimplified physical model can introduce **systematic bias**.

Classification of measurement errors



Random errors (Type A)



- Due to unpredictable fluctuations.
- Change in magnitude and sign.
- Average to zero over many measurements.
- Reduce by increasing the number of measurements.

Systematic errors (Type B)



- Due to a consistent cause.
- Lead to a bias (offset or scale error).
- Do not average to zero.
- Reduce by improving the method/instrument or applying a **correction**.



Key message:

Random errors affect precision; systematic errors affect accuracy.
Both must be considered to obtain a reliable measurement result.

Sources of measurement errors

Random errors (Type A)



Instrumental noise

Intrinsic fluctuations of the instrument or its electronics.



Short-term environmental fluctuations

Small changes in temperature, pressure, humidity, etc.



Observer variations

Differences in reading, reaction time, or estimation.



Random sampling variations

Natural scatter due to the measured quantity itself.

Systematic errors (Type B)



Calibration errors

Instrument not properly calibrated or drift from its calibration.



Environmental influences

Constant temperature gradient, vibrations, electromagnetic interference, etc.



Method or procedure errors

Approximate methods, assumptions, or simplifications.



Instrument imperfections

Nonlinearity, hysteresis, friction, zero offset, resolution limits, etc.



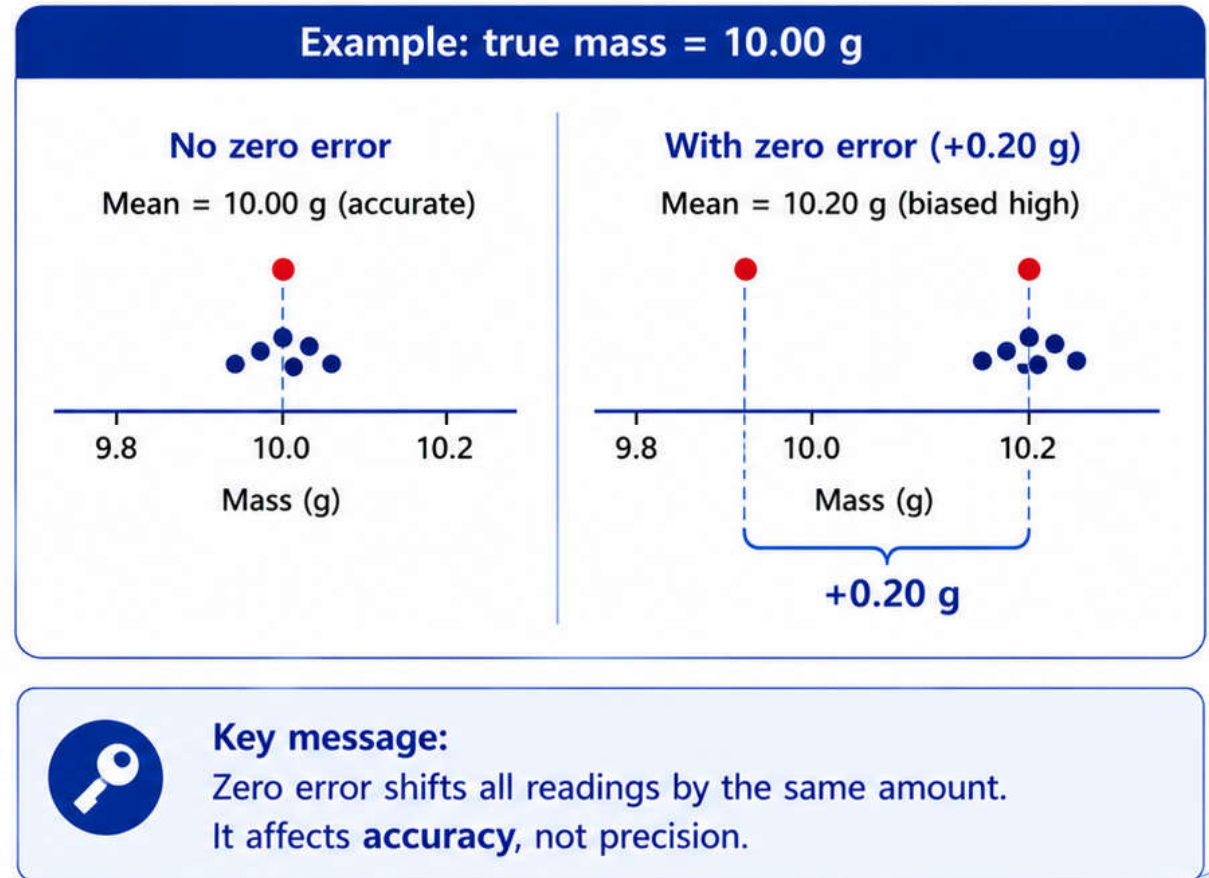
Key message:

Random errors cause scatter and limit precision; systematic errors cause bias and limit accuracy. Identify the sources to reduce their impact!

Example: zero error

A scale is misadjusted so that it reads **+0.20 g** even when the true mass is 0.00 g. This causes a systematic error (zero error).

- ➔ Every measurement is too high by **+0.20 g**.
- ➔ Repeated measurements give consistent results, but they are all **biased**.
- ➔ The average is not the true value.
- ➔ The error must be detected (e.g., by zero check) and corrected.



Where should we look for systematic errors?

Systematic errors originate from specific, consistent causes in the measurement process. We should look where bias can be introduced.

- In the **instruments** and their **calibration**.
- In the **environment** affecting the measurement.
- In the **method, procedure**, and **assumptions**.
- In the **materials, standards**, and **reagents** used.
- In the **operator**: reading technique and behavior.
- In the **data processing** and calculations.
- In the **model** used to interpret the data.

Typical sources and examples		
	Instrument (calibration)	Zero error, scale factor error, drift, misalignment, hysteresis.
	Environment	Temperature, pressure, humidity, vibration, electromagnetic fields, drafts, lighting.
	Method and procedure	Incomplete reaction, wrong timing point, contamination, losses, evaporation.
	Materials and standards	Impure reagents, degraded standards, incorrect reference values.
	Operator	Parallax (reading angle), reaction time, force applied, personal habits.
	Data processing and calculations	Rounding, wrong formula, unit conversion, transcription errors.
	Model and assumptions	Simplifying assumptions, neglected effects, inappropriate model for the system.



Key message:

Systematic errors come from identifiable sources.

Knowing where to look is the first step to **detect and correct** them.

Model error as systematic error



Definition

Model error occurs when the theoretical model used to interpret data is an imperfect representation of reality. It produces a **bias** because the model's assumptions are not fully valid.

Model vs. reality

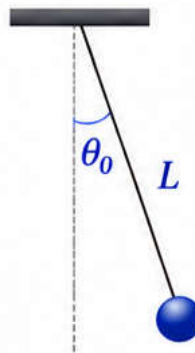
Ideal (model)	Real (experiment)
 Simple harmonic motion	Small deviations from SHM (large angles, damping)
 Length L to pivot	Effective length to center of mass is slightly different
 No air resistance	Air drag present
 Frictionless pivot	Small friction at pivot



Key message:

Even with perfect measurements, using an **imperfect model** introduces a **systematic error**. Improving the model or applying corrections reduces the bias.

Example: pendulum and the simple model



Simple model assumptions:

- Small angles ($\sin \theta \approx \theta$)
- Massless, inextensible string
- Point mass bob
- No air resistance
- Frictionless pivot

Real pendulum violates these assumptions → **systematic error** in g .

Effect on g

- L measured to the pivot, not to the center of mass.
- Large amplitude → period increases → g underestimated.
- Air resistance and friction → period increases → g underestimated.



Because the model is imperfect, the result is **systematically biased**.

How to reduce model error

- ✓ Use a more complete model.
- ✓ Apply correction factors.
- ✓ Keep assumptions valid (e.g., small angles).

When systematic error cannot be eliminated

In some cases the source of the systematic error cannot be removed completely. We can only estimate and account for its effect in our results.



1. Quantify the error

Estimate the size (and possibly direction) of the systematic error.



2. Apply a correction

Use an appropriate correction factor or an alternative method. If the sign is unknown, base the estimate on physical reasoning.



3. Include in the uncertainty

If full correction is not possible, include an uncertainty contribution due to the unavoidable systematic error.



4. State clearly

Be transparent about the limitation and the estimated effect.

Example: resistance measurement with systematic error

1. Measured values

A mmeter has a zero error of $+2 \Omega$. Measured values:

Run	$R_{\text{meas}} (\Omega)$
1	11.8
2	12.1
3	12.0
4	12.2
5	12.0

2. Systematic error cannot be fully eliminated

- Zero error remains (e.g., due to instrument imperfection).
- We assume the true value is 2.0Ω lower than measured value ($s = 2 \Omega$).

$$R_{\text{true}} = R_{\text{meas}} - 2.0 \Omega$$

Thus the corrected mean:

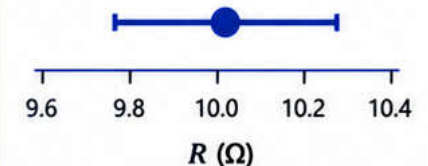
$$\bar{R} = 10.0 \Omega$$

(theoretical value)

3. Residual uncertainty

If it is not possible to remove the error completely, include a systematic uncertainty: $\Delta_{\text{syst}} = 0.2 \Omega$.

Distribution of results



— Mean value $\pm$ random uncertainty
 $\pm$ systematic uncertainty



Key message:

When a systematic error cannot be eliminated, we must **estimate its effect, correct for it if possible,** and include it in the uncertainty. If it cannot be corrected, we still report the result with the estimated systematic uncertainty.

Changing the experimental conditions



Key idea: If a systematic error depends on certain conditions, changing those conditions can help identify or reduce its effect.

Why do it?

- Systematic errors often depend on controllable factors (e.g., temperature, pressure, instrument range).
- Changing conditions can reveal trends that indicate the presence and size of a bias.
- It can help separate systematic effects from random noise.

How to do it?

- 1 Identify which conditions may influence the measurement.
- 2 Plan a series of measurements while varying one factor at a time.
- 3 Analyze how the results change with the condition.
- 4 Use the results to correct for or reduce the systematic error.

Examples of changing conditions

Condition changed	Example (pendulum g experiment)	What we look for	What it can reveal
 Temperature	Measure g at different laboratory temperatures.	Does the result vary with T ?	Thermal expansion of the string or stand.
 Amplitude (angle)	Measure g at different initial angles (e.g., 2° , 5° , 10°).	Does the period increase with angle?	Large-angle effect (model error).
 Length	Use different pendulum lengths (short, medium, long).	Does g depend on L ?	Length-dependent bias (e.g., pivot effect).
 Timing method	Time $N = 10, 20, 50, 100$ oscillations.	Does the result depend on N ?	Reaction-time bias becomes smaller as N increases.
 Instrument settings	Use different instruments or ranges (e.g., different stopwatches).	Do different instruments give different results?	Calibration or scale factor error.
 Environment	Repeat on different days, humidity, air currents.	Does the result change with environment?	Air resistance, drafts, or other external influences.



Key message:

Changing experimental conditions is a powerful way to **diagnose and reduce systematic errors**. If a result depends on a condition that should not affect it, a systematic error is likely present.

Same quantity, different experimenters



Key idea: If different experimenters obtain different results for the same quantity under nominally the same conditions, a systematic error may be present in one or more sets of measurements.

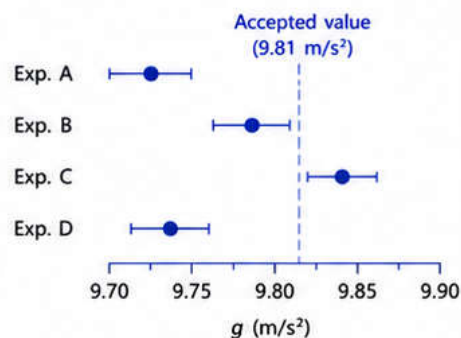
Why do it?

- Different habits, techniques, or equipment can introduce biases.
- Comparing results helps reveal hidden systematic errors.
- It is a powerful cross-check for reliability.

How to do it?

- 1 Several experimenters measure the same quantity following a common protocol.
- 2 Each records their individual results and uncertainty estimates.
- 3 Compare the results and uncertainties.
- 4 Investigate any significant differences.
- 5 Identify possible causes and correct if possible.

Example (measuring g)



Interpreting the comparison

Situation	What it suggests	What to do
All results agree within uncertainties	No significant systematic differences detected.	Combine results (e.g., weighted average).
All results are offset in the same direction	Common systematic error (e.g., model or environment).	Look for shared cause; apply correction if known.
One result is significantly offset from others	Experimenter-specific systematic error likely.	Investigate that setup or technique.
Results are scattered beyond uncertainties	Unrecognized systematics or poor control of conditions.	Improve procedure; check equipment and conditions.
Differences correlate with a known factor	Systematic error depends on that factor.	Control that factor or correct for its effect.



Interpretation:

Agreement within uncertainties increases confidence. Disagreements are a clue to systematic effects.



Key message:

Comparing measurements of the same quantity by different experimenters helps **uncover systematic errors** that might be hidden within a single set of results.

Same quantity, different methods



Key idea: Measuring the same quantity with different methods helps identify systematic errors that may be specific to one method.

Why do it?

- Different methods rely on different assumptions and instruments.
- A systematic error in one method may not appear in another.
- Consistent results across methods increase confidence in the value.

How to do it?

- 1 Choose two or more independent methods to measure the same quantity.
- 2 Ensure that the methods are based on different principles or assumptions.
- 3 Measure carefully with each method and estimate uncertainties.
- 4 Compare the results and their uncertainties.
- 5 Investigate and address any significant differences.



Key message:

Using different methods to measure the same quantity is one of the most powerful ways to **detect** and **understand** systematic errors.

Example: measuring the acceleration due to gravity, g

Method	Principle	Typical result (m/s^2)	Possible systematic errors	What comparison can reveal
 1. Simple pendulum	Period of small oscillations: $T = 2\pi\sqrt{L/g}$	9.81 ± 0.10	• Length measurement (pivot to CM) • Large-angle effect	Length or angle systematics not present in other methods.
 2. Free fall	Distance fallen in time t : $s = \frac{1}{2}gt^2$	9.74 ± 0.15	• Reaction time • Air resistance • Release mechanism	Timing or air-resistance biases affecting only this method.
 3. Inclined plane	Acceleration down incline: $a = g\sin\theta$	9.79 ± 0.12	• Angle measurement • Friction	Friction or angle calibration systematics.
 4. Spring-mass oscillator	Period of mass-spring oscillation: $T = 2\pi\sqrt{m/k}$	9.77 ± 0.13	• Mass measurement • Spring constant calibration	Mass or spring constant biases unique to this method.



Interpretation: If results agree within uncertainties → no significant systematic errors between methods.
If not → investigate which method(s) are affected and why.

Comparing two measurements



Key idea: When we have two independent measurements of the same quantity, we compare how far apart they are relative to their uncertainties.

Two independent measurements of the same quantity X

$X_A \pm u_A$
Measurement A

$X_B \pm u_B$
Measurement B

Difference (absolute):

$$D = |X_A - X_B|$$



Combined standard uncertainty of the difference:

$$u_D = \sqrt{u_A^2 + u_B^2}$$

(valid if the two uncertainties are independent)

The key question:

Is D compatible with the stated uncertainties?

We compare D with the combined uncertainty u_D .



Yes, compatible
($D \leq k u_D$)

The difference is small enough to be explained by random effects and the stated uncertainties.

→ **Results are consistent.**



No, not compatible
($D > k u_D$)

The difference is too large to be explained by the stated uncertainties.

→ **Possible systematic effect or missing uncertainty component.**



Typical choice: $k = 2$ (about 95% level)

Other choices ($k = 1, 2, 3$) are also used depending on context.



Key message: Quantify the difference between two measurements and compare it with their combined uncertainty. This simple test is a powerful tool for revealing hidden systematic errors.

Visual consistency test



Key idea: A simple visual comparison of the uncertainty ranges often reveals whether two measurements are consistent with each other.



Central value $\pm$ uncertainty

Overlap of uncertainty ranges
→ compatible (consistent)

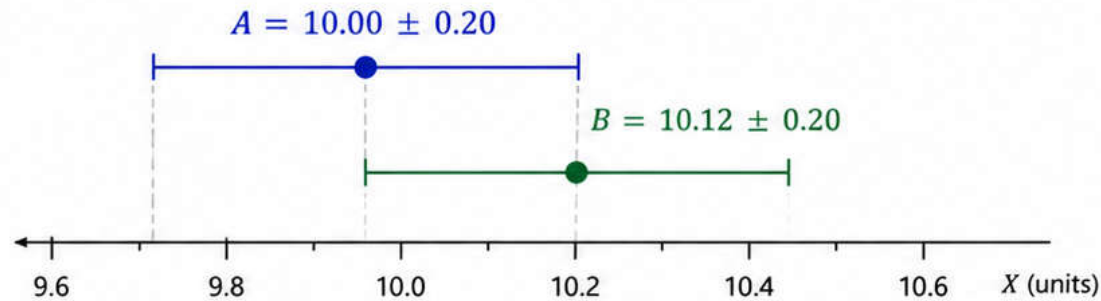
No overlap → may indicate a systematic effect

Case A:

Overlapping uncertainty ranges



CONSISTENT



The uncertainty ranges overlap.

→ The difference can be explained by random effects and the stated uncertainties.

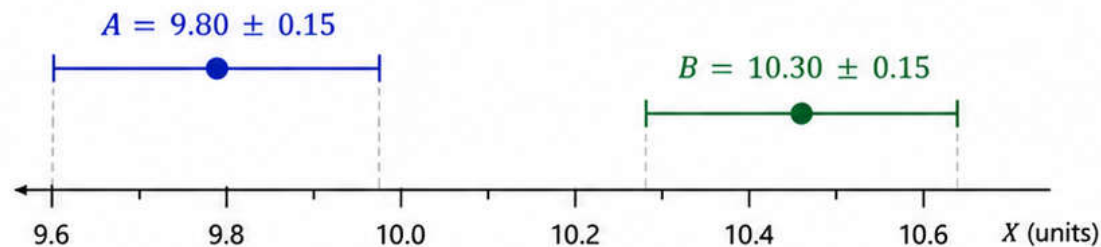
→ The results are **consistent**.

Case B:

No overlap of uncertainty ranges



POSSIBLE SYSTEMATIC EFFECT



The uncertainty ranges do not overlap.

→ The difference is too large to be explained by random effects and the stated uncertainties.

→ There may be a **systematic effect** (or underestimated uncertainties, or an incorrect model).



Note: Overlap does not guarantee that systematic errors are absent. It only means that no significant inconsistency is visible.

Consistent vs. inconsistent measurements



Key idea: Comparing independent measurements helps us judge whether our uncertainty model is adequate.

When measurements are consistent



The difference between results is compatible with the stated uncertainties.

→ Our understanding of random effects and systematic effects may be adequate.

When measurements are inconsistent



The difference is too large to be explained by the stated uncertainties.

→ Something is missing from the uncertainty model.



Key messages:

Consistency **does not prove** that systematic errors are absent.

Inconsistency **is evidence** that something is missing from the uncertainty model.

Possible causes of inconsistency



1. Systematic effects

Unaccounted-for biases (e.g., calibration offsets, environmental influences, approximation errors).



2. Underestimated uncertainties

Some uncertainty components were missed, or their size was underestimated.



3. Incorrect model

The underlying physical model or assumptions may be incomplete or wrong.



What to do next?

- 1 Re-examine the experiment, instruments, and data analysis.
- 2 Look for missing or unaccounted sources of uncertainty.
- 3 Refine the model or method and, if possible, repeat the measurements.



Important: Consistency increases our confidence, but it does not certify that all systematic errors are absent. Inconsistency is a **valuable clue** that leads us to better experiments and better science.

How do we detect systematic errors?



Key idea: Systematic errors are not revealed by a single measurement. We detect them by a structured comparison of independent results under different conditions.



Principle:

Change something that could influence a systematic effect, then check whether the result changes in a way that exceeds the stated uncertainties.



Goal

Through repetition, variation, and comparison, we reveal hidden systematic effects and increase the reliability of our measurements.



1. Repeat measurement

Perform several measurements under the same conditions.



2. Same conditions

Evaluate random effects

Analyze the scatter of results. This estimates random uncertainty.



3. Change conditions

Change one or more conditions that could affect a systematic effect (e.g., instrument setting, position, environment, operator, method).



4. Compare independent results

Use different methods, instruments, setups, times, or laboratories when possible.



5. Consistent / Inconsistent?

Are the independent results compatible with the stated uncertainties?



Consistent

No evidence of significant systematic effect (yet).



Inconsistent

Evidence that something is missing from the uncertainty model.



6. Investigate: instrument + method + model

Identify possible causes and refine the experiment, uncertainty estimates, or the physical model.

- Check calibration and alignment
- Look for unaccounted influences
- Reconsider assumptions and theory
- Improve method or model
- Repeat and verify



Takeaway: Systematic errors are detected by a cycle of repeat → change → compare → investigate. This is the heart of good experimental practice.

Combining measurements



Goal: Obtain the best possible estimate of a quantity from several independent measurements.

Suppose we have two independent measurements of the **same** quantity X .

$$X_A \pm u_A$$

Measurement A

$$X_B \pm u_B$$

Measurement B



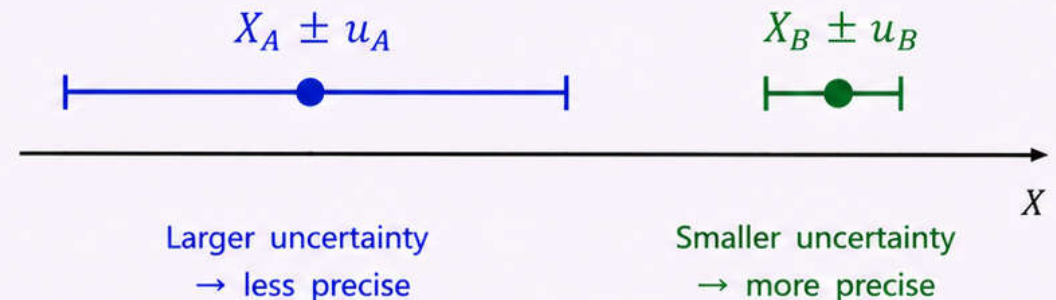
Question:
Should both measurements contribute **equally**?



No! If one measurement is much more precise (smaller uncertainty), it should influence the result more.

Why this matters?

- Measurements have different uncertainties.
- More precise measurements should have more influence on the final result.
- We need a principled way to combine them.



Idea of weighting



Key idea: More precise measurements → **larger weight**.

We assign a weight to each measurement that is inversely proportional to its variance.

Interpretation:

- **Small** uncertainty u_i (high precision) → **large** weight w_i .
- **Large** uncertainty u_i (low precision) → **small** weight w_i .

Weights are chosen as

$$w_i = \frac{1}{u_i^2}$$

where u_i is the standard uncertainty of measurement i .



Consequence:

Measurements with larger weights pull the final result more strongly toward their values.

Visual example (two measurements of the same quantity X)



Conclusion: In combining measurements, we let the more precise results have a stronger influence.

Weighted mean



Goal: Combine several independent measurements of the same quantity into a single best estimate.

Suppose we have N independent measurements of the same quantity X :

$$X_1 \pm u_1$$

$$X_2 \pm u_2$$

...

$$X_N \pm u_N$$



Weighted mean:

$$X_w = \frac{\sum_{i=1}^N w_i X_i}{\sum_{i=1}^N w_i}$$

with weights

$$w_i = \frac{1}{u_i^2}$$

Interpretation

- Each measurement X_i contributes in proportion to its weight w_i .
- More precise measurements (smaller u_i) have **larger weights** and thus more influence on the result.



Special case: If all uncertainties are equal ($u_i = u$ for all i), then all weights are equal and X_w reduces to the ordinary mean:

$$X_w = \frac{1}{N} \sum_{i=1}^N X_i$$



Takeaway: The weighted mean uses the reliability (precision) of each measurement. Better measurements count more.

Uncertainty of the weighted mean



Goal: Determine the uncertainty (standard uncertainty) of the weighted mean.

For N independent measurements $X_i \pm u_i$ ($i = 1, 2, \dots, N$)

Weighted mean:

$$X_w = \frac{\sum_{i=1}^N w_i X_i}{\sum_{i=1}^N w_i}$$

Weights:

$$w_i = \frac{1}{u_i^2}$$



Uncertainty of the weighted mean:

$$u(X_w) = \frac{1}{\sqrt{\sum_{i=1}^N \frac{1}{u_i^2}}}$$

Equivalent form:

Since $w_i = \frac{1}{u_i^2}$, we have

$$u(X_w) = \frac{1}{\sqrt{\sum_{i=1}^N w_i}}$$



Important note

The above formula is rigorously derived under the following assumptions:

- Each measurement error is random, independent, and normally distributed.
- u_i is the standard deviation (1σ) of measurement i .

Under these conditions, $u(X_w)$ is the standard uncertainty (1σ) of the weighted mean.

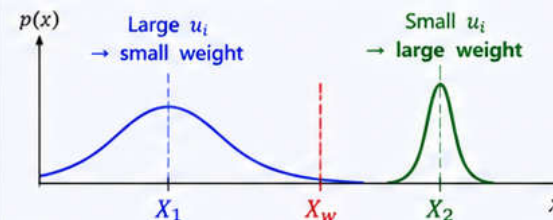


Beyond the assumptions

If the errors are not normally distributed, or if uncertainties are correlated, non-standard, or estimated by other methods, then the formula for $u(X_w)$ is not exact.

In such cases, the weighted mean and its uncertainty are still useful as good approximations, but they should be applied with judgement.

Why this makes sense (visual intuition)



The result is pulled toward the more precise measurement.



Takeaway: The uncertainty of the weighted mean decreases as more precise (smaller u_i) measurements are included. Better (more precise) measurements make the final result both more accurate and more precise.

26 Randomization of systematic effects

Measurements performed with:



different instruments



different observers



different methods



different laboratories



Systematic effects

may vary
between
measurements.



Their distribution

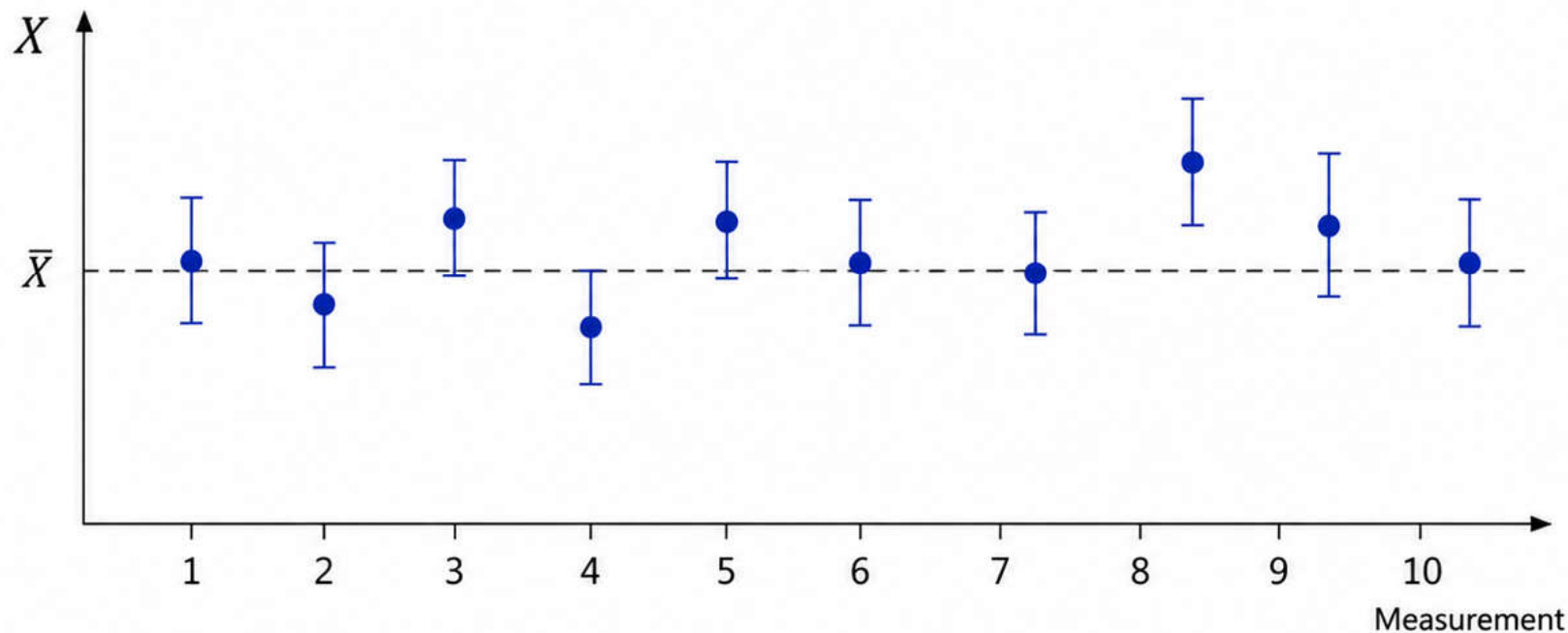
can sometimes be
treated
statistically.



If the conditions are varied from measurement to measurement, systematic effects may be **randomized**.

27 Distribution of independent results

Suppose we have N independent measurements of the same quantity X :



Notation

$X_1, X_2, \dots, X_N$
independent results
(with individual
uncertainties u_i)

Mean of the results

$$\bar{X} = \frac{1}{N} \sum_{i=1}^N X_i$$



The spread (dispersion) of the independent results around the mean $\bar{X}$ contains information about the **systematic contribution**.

28 Are the measurements consistent?

To assess the **reliability** of a set of independent measurements, compare two quantities:

1. **Uncertainty expected
from individual measurements**

(from reported uncertainties)



u_{expected}

VS.

2. **Observed dispersion between
independent measurements**

(standard deviation of the results)



u_{observed}

If

$$u_{\text{observed}} > u_{\text{expected}}$$



The observed spread is larger than expected.
This indicates **systematic effects** and/or
underestimated individual uncertainties.

If

$$u_{\text{observed}} \approx u_{\text{expected}}$$



The observed spread is consistent with what is expected
from the individual uncertainties.
No significant additional systematic effects detected.



A quantitative check can be performed using the χ^2 test or the **Birge ratio** (see later in the course).

29 Important limitation



**RANDOMIZATION DOES NOT
REMOVE SYSTEMATIC EFFECTS.**

It allows their influence to be
ESTIMATED STATISTICALLY.



Randomization can reveal and quantify the presence of systematic effects, but it **cannot eliminate** or **correct** them.

30 Type A and Type B evaluation

Type A

statistical evaluation



repeated measurements



sample statistics



experimental distributions

Type B

non-statistical evaluation



calibration data



resolution



manufacturer specification



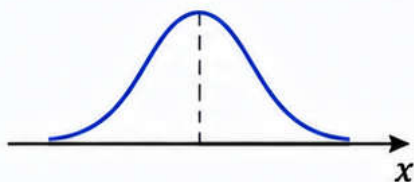
Both **Type A** and **Type B** evaluations are used.
Together they provide a **complete description** of measurement uncertainty.

31 Standard uncertainty

Unified idea: express every uncertainty contribution in the same way.

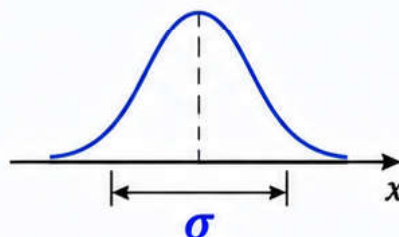
1

Assign a **probability distribution** to each uncertainty contribution.



2

Represent its **width** by a **standard deviation**.



3

This standard deviation is the **standard uncertainty u** .

$$u = \sigma$$



The standard uncertainty u is a measure of the **dispersion** of values that can reasonably be attributed to the quantity.



The standard uncertainty provides a **common basis** for combining different uncertainty contributions.

32 Combining uncertainty contributions

When the uncertainty contributions are **independent**, the combined standard uncertainty is obtained by the quadratic sum (root-sum-square).

If a measurand Y depends on several **independent** input quantities

$$Y = f(X_1, X_2, \dots, X_n)$$

with standard uncertainties $u_1, u_2, \dots, u_n$, then the combined standard uncertainty of Y is

$$u_c = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2} = \sqrt{\sum_{i=1}^n u_i^2}$$

where u_i is the standard uncertainty associated with the i -th input quantity.

Key points

- **Independence** is essential.
- **Variances** add, standard uncertainties are combined by the square root.
- The result has the **same unit** as the measurand.



Note:

If contributions are **correlated**, **covariance** terms must be included.



This rule will also be used in **uncertainty propagation** for indirect measurements, which we will return to in a later part of the course.



For the general case with **correlations**: $u_c^2 = \sum_{i=1}^n u_i^2 + 2 \sum_{i < j} u_i u_j \rho_{ij}$, where ρ_{ij} is the **correlation coefficient**.

34 Reporting a measurement

Standard form:

$$X = X_0 \pm u$$

where X_0 is the best estimate of the measurand and u is its standard uncertainty.

Always include the unit.

Example:

$$L = (125.4 \pm 0.3) \text{ mm}$$

best estimate
(rounded value)

standard
uncertainty

unit

Rules for reporting

1



Uncertainty: usually reported with **1–2 significant digits**.

2



Value: rounded to the same **decimal position** as the uncertainty.

3



Unit: always include the unit.



Good reporting makes results **clear**, **consistent**, and **comparable** across experiments and laboratories.

35 Alternative notation

An alternative (compact) notation is often used:

$$L = 2.357(25) \text{ m}$$

The digits in parentheses give the uncertainty in the last digits of the reported value.



General rule:

If the last digits written are in the k -th decimal place, the number in parentheses gives the uncertainty in the same k -th decimal place.

Here: (25) $\rightarrow$ uncertainty in the last two digits $\rightarrow$ **0.025 m**



Thus, $L = 2.357(25) \text{ m}$ means **$L = (2.357 \pm 0.025) \text{ m}$**

Meaning:

$$L = 2.357(25) \text{ m}$$

=

$$L = (2.357 \pm 0.025) \text{ m}$$

The uncertainty affects only the last digits shown.



Example: physical constants

Physical constants are often reported using this notation.

Example (speed of light in vacuum):

$$c = 299\,792\,458(0) \text{ m/s}$$

means $c = (299\,792\,458 \pm 0) \text{ m/s}$



Key point:

This notation is unambiguous, compact, and widely used in scientific publications, tables of constants, and data sheets.

Examples:

- $g = 9.80665(10) \text{ m/s}^2$
- $h = 6.626\,070\,15(23) \times 10^{-34} \text{ J s}$

36

Relative uncertainty

The **relative uncertainty** (or fractional uncertainty) is

$$u_r = \frac{u(X)}{|X|}$$

where X is the measured value and $u(X)$ its standard uncertainty.

Relative uncertainty in percent:

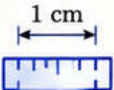
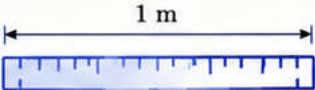
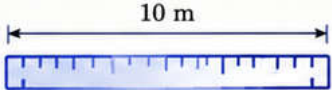
$$u_r(\%) = \frac{u(X)}{|X|} \times 100\%$$

Why relative uncertainty matters

The same absolute uncertainty can represent very different measurement quality depending on the size of the measured value.

Example (from Fornasini):

Absolute uncertainty: $u = 1$ mm in all cases.

Case 1: $L = 1$ cm	Case 2: $L = 1$ m	Case 3: $L = 10$ m
		
$u = 1$ mm	$u = 1$ mm	$u = 1$ mm
$u_r = \frac{1 \text{ mm}}{10 \text{ mm}} = 0.10 = 10\%$	$u_r = \frac{1 \text{ mm}}{1000 \text{ mm}} = 0.001 = 0.1\%$	$u_r = \frac{1 \text{ mm}}{10000 \text{ mm}} = 0.0001 = 0.01\%$
Poor	Good	Excellent



Relative uncertainty is **dimensionless**. It allows meaningful comparison of measurement quality at different scales.

Typical interpretation

- $u_r < 1\%$: high-quality measurement
- $1\% \leq u_r < 5\%$: acceptable
- $u_r \geq 5\%$: low quality (check the measurement!)



Notes

- Relative uncertainty is often more informative than absolute uncertainty.
- Always report both the result in **standard form** and, when relevant, the **relative uncertainty**.

Repeatability, reproducibility and accuracy



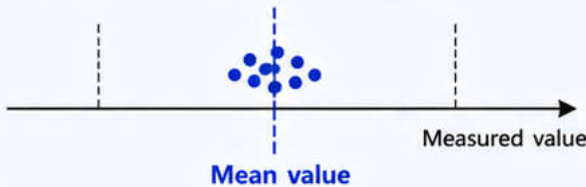
These three terms describe **different aspects** of measurement quality. They answer different questions.

Repeatability

- Same method
- Same instrument
- Same observer
- Short time interval



How close are repeated results?

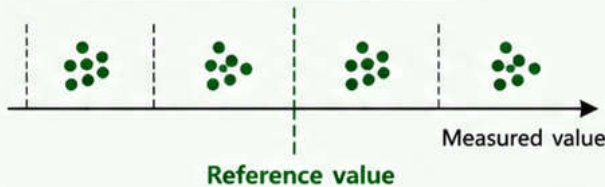


Reproducibility

- Different conditions
- Different laboratories / instruments / observers



How robust is the result?

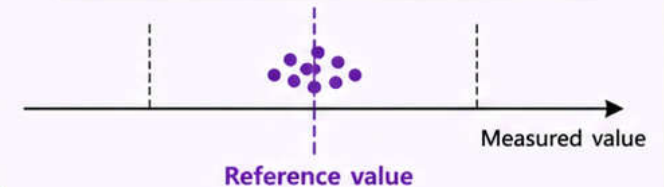


Accuracy

- Comparison with an accepted / reference value



How close is the result to the accepted/reference value?



High repeatability → small spread of repeated results under the same conditions.

High reproducibility → consistent results under different conditions and by different people/equipment.

High accuracy → results are close to the accepted/reference value (small bias).





This lecture introduced the concepts and tools for evaluating and reporting **measurement uncertainty**. Here are the main points we covered.



1. Nature of uncertainty

- Every measurement has uncertainty.
- Two main types: random (statistical) and systematic.
- Uncertainty cannot be eliminated, only reduced.



2. Sources of uncertainty

- Instrument limitations (resolution, calibration, drift).
- Method and model assumptions.
- Environmental influences.
- Observer (human) effects.



3. Estimating uncertainty

- Use of repeated measurements to assess random uncertainty.
- Type A: statistical methods.
- Type B: other information and engineering judgment.



4. Combining results

- Results with different uncertainties are combined using appropriate weighting.
- Weighted mean gives the best estimate.



5. Uncertainty budget

- Identify all significant contributions to uncertainty.
- Express as standard uncertainties.
- Combine using the root-sum-of-squares.



6. Reporting and quality

- Report results in standard form with unit and uncertainty.
- Relative uncertainty allows meaningful comparison.
- Quality: repeatability, reproducibility, accuracy.



Careful evaluation, proper combination, and clear reporting of uncertainty are essential for **reliable** and **comparable** measurement results.

39 Take-home messages



1. Every measurement has **uncertainty**.



2. Repetition reduces **random uncertainty** but does not eliminate **systematic error**.



3. Changing **experimental conditions** helps reveal **systematic effects**.



4. Measurements with **different uncertainties** should be combined using **appropriate weighting**.



5. Different uncertainty contributions can be expressed as **standard uncertainties** and **combined consistently**.



6. A measurement result is **incomplete** without its **uncertainty**.



Next lecture → *Uncertainty propagation in indirect measurements*

