



Measurement Uncertainty in Direct Measurements I

From Measurement Resolution to Statistical Uncertainty



Sources of uncertainty



Random fluctuations



Statistical description of data



Distribution of sample means



Lecture 4
Part 1



90 minutes



Based on:
The Uncertainty in Physical Measurements
Paolo Fornasini, Springer, 2008 (Ch. 4.1–4.3)



Department of Optical
and Quantum Electronics



From the Previous Lecture

In the previous lectures we learned how to present experimental data properly:



Significant digits

Report only meaningful digits.



Rounding

Reflect the precision of the result.



Tables

Identify quantities, units and uncertainties.



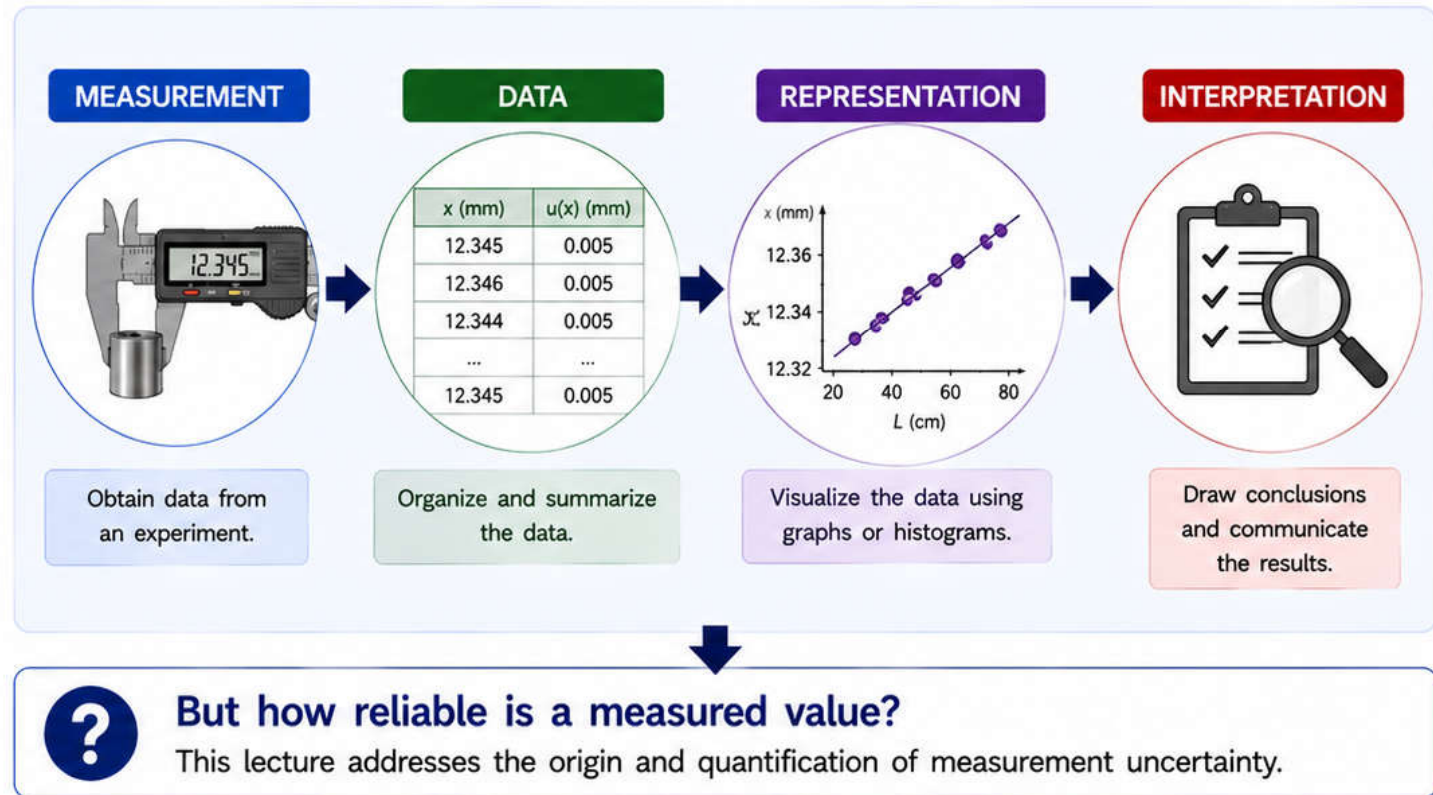
Graphs

Visualize relationships and trends using appropriate scales.



Histograms

Reveal the distribution of repeated measurements.



Good experimental science requires not only correct **measurements**, but also correct communication of their **uncertainty**.





What Will We Learn Today?

This lecture is the first part of our study of measurement uncertainty in direct measurements.

1



Where does measurement uncertainty come from?

We will identify the main sources of uncertainty in physical measurements.

2



How do repeated measurements fluctuate?

We will study random fluctuations and how they affect measurement results.

3



How can we describe their distribution?

We will learn how to use statistics and probability distributions to model measurement data.

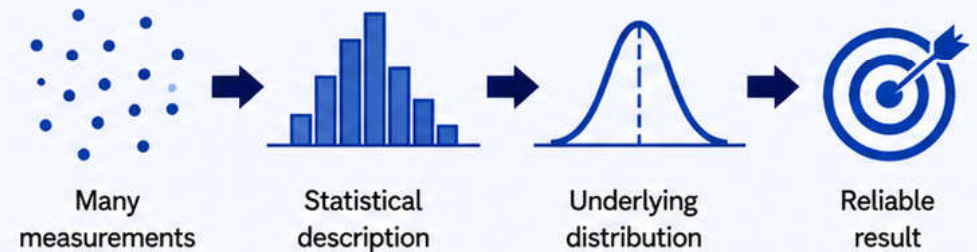
4



How does repetition reduce uncertainty?

We will see how averaging multiple measurements leads to a more precise result.

From Individual Measurements to Reliable Results



Understanding and quantifying **uncertainty** are essential for trustworthy experimental science.



*Good measurements lead to good data. → Understanding uncertainty leads to **good science**.*





A Measurement Is Never an Exact Number

$$X = \underbrace{\text{measured value}} \pm \underbrace{\text{uncertainty}}$$



True value
(unknown)

The true value of a measurand is the exact value that would be obtained by an ideal measurement.

It can never be exactly known.



Measured value
(observable)

The result we obtain in an experiment by a real measurement.

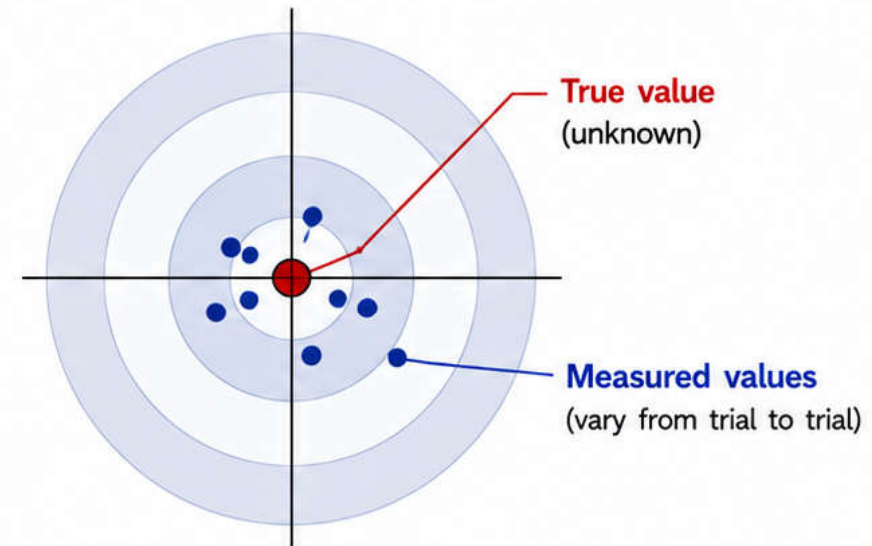
It is only an estimate of the true value.



Uncertainty
(quantifies our knowledge)

A parameter associated with the measured value that characterizes the possible deviation from the true value.

It expresses the limits of our knowledge.



A reported measurement is **meaningful** only when accompanied by an **uncertainty**.



Measurements are never perfect, but they can be reliable.



Quantifying uncertainty allows us to judge reliability and compare results.

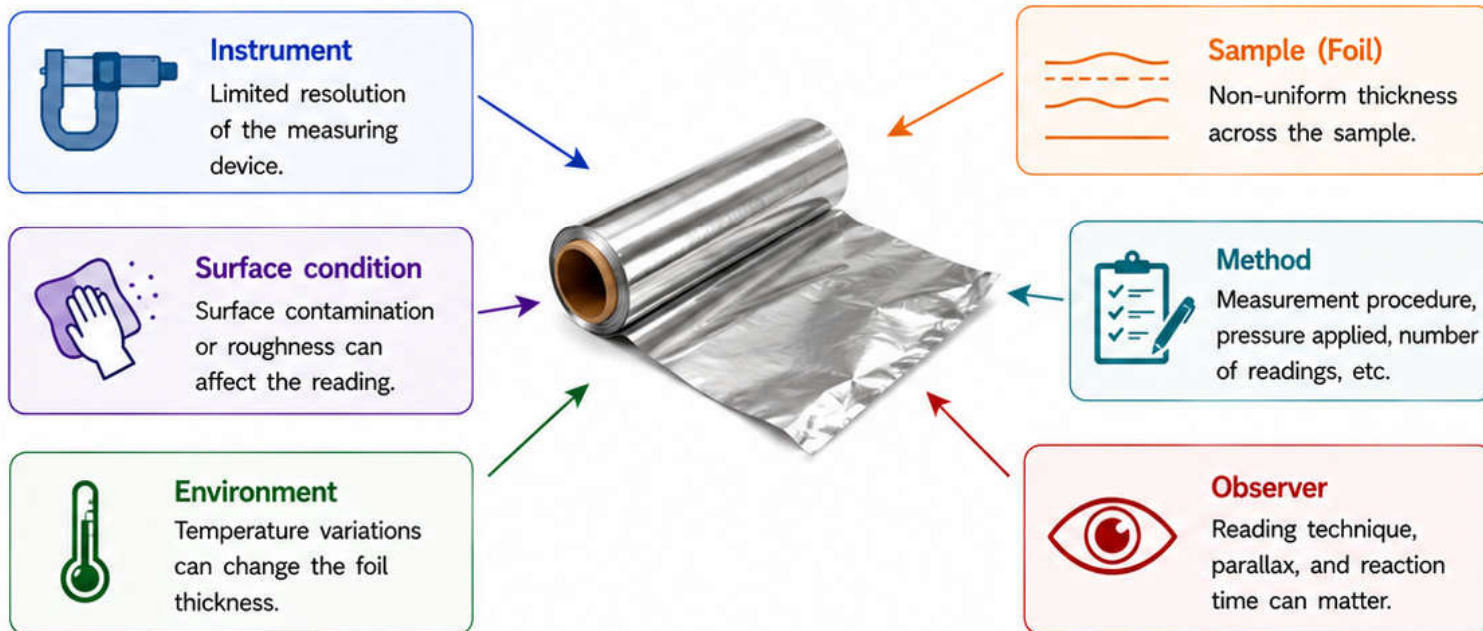


Good science requires not only measurements, but also honesty about their limitations.



Example: Measuring the Thickness of Aluminium Foil

A single measurement is influenced by many factors. Recognizing them helps us understand and quantify the uncertainty.



One measured quantity – many uncertainty sources

Understanding these sources allows us to:

- ✓ identify the dominant contributors,
- ✓ improve the measurement procedure,
- ✓ estimate the uncertainty more realistically.



Uncertainty cannot be eliminated, but it can be **understood**, **reduced**, and **quantified**.



The total uncertainty of a measurement is the **combined effect** of all relevant sources.



Three Main Contributions to Measurement Uncertainty

Measurement uncertainty arises from different sources. They can be grouped into three main categories.

1



Measurement resolution

The finite resolution of the measuring instrument limits how precisely a quantity can be read. (Quantization of the indicated value.)

2



Random fluctuations (random error)

Unpredictable variations in repeated measurements caused by many small, uncontrollable influences. They lead to scatter in the results.

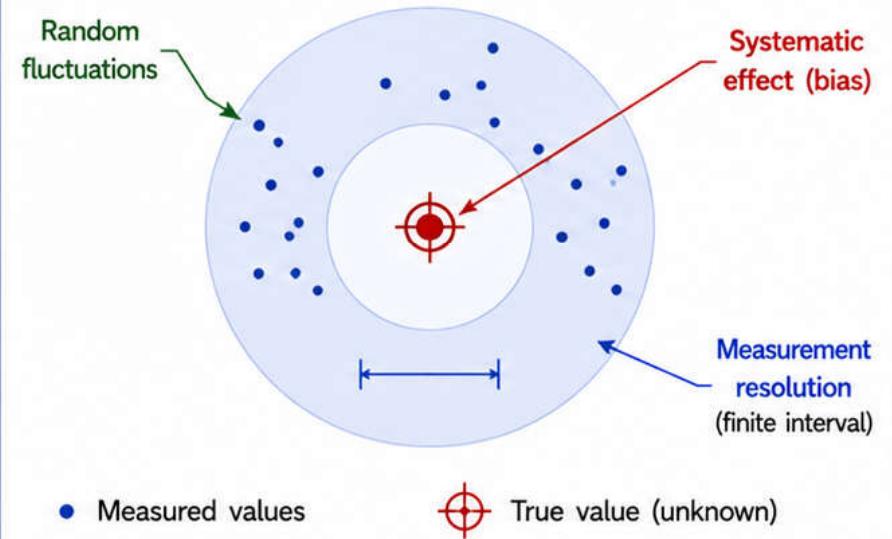
3



Systematic effects (systematic error)

Reproducible biases that shift all results in the same direction due to imperfections in instruments, methods, or conditions.

Where Do the Uncertainties Come From?



This lecture focuses mainly on the **first two** contributions. Systematic effects will be discussed in later lectures.



All three contributions can affect the final result, often at the same time. → Our goal is to understand and quantify their impact.



Measurement Resolution

The **resolution** of the measuring instrument sets a fundamental limit to the precision of the indicated value.

Instrument resolution

$$\Delta x_{\text{inst}}$$



Smallest change that the instrument can distinguish or display.

≠

Measurement resolution

$$\Delta x_{\text{meas}}$$

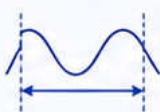


Smallest change in the measurand that can be resolved by the entire measurement procedure.

Example:
Timing a
Periodic Signal

Measure one period

$$\Delta t_{\text{inst}} = 0.50 \text{ s}$$

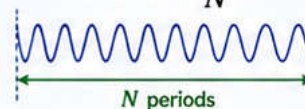


$$\Delta t_{\text{meas}} \geq \Delta t_{\text{inst}}$$



Measure N periods

$$\Delta t_{\text{meas}} \geq \frac{\Delta t_{\text{inst}}}{N}$$



They Are Not Necessarily the Same



May be equal: $\Delta x_{\text{meas}} = \Delta x_{\text{inst}}$



May be larger: $\Delta x_{\text{meas}} > \Delta x_{\text{inst}}$

e.g., reading a scale with parallax, influence of the environment, poor technique.



May be smaller: $\Delta x_{\text{meas}} < \Delta x_{\text{inst}}$

e.g., timing many periods of a signal.



Key idea:

The measurement procedure itself can improve the achievable resolution, but never below the fundamental limit set by physics and noise.



When a measurement is performed with a measurement resolution Δx_{meas} , the reported value X can lie anywhere within the corresponding resolution interval. All values in this interval are **equally plausible** (assuming no other information).





Can We Measure More Precisely Than the Instrument Resolution?

The measurement resolution Δx_{meas} depends not only on the instrument, but also on the measurement procedure.



Example

Measuring the period T of a periodic signal using a digital stopwatch.

 Δx_{inst}

Instrument resolution

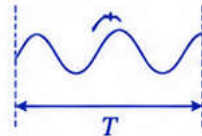
(smallest displayed change)

 Δx_{meas}

Measurement resolution

(effective resolution of the measurement)

1 Measure one period



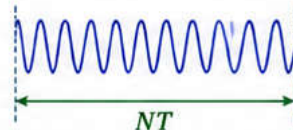
$$\Delta x_{\text{inst}} = 0.5 \text{ s}$$



$$\Delta x_{\text{meas}} \geq \Delta x_{\text{inst}}$$

The measured time cannot be more precise than the instrument resolution.

2 Measure N periods



$$\Delta x_{\text{inst}} = 0.5 \text{ s}$$



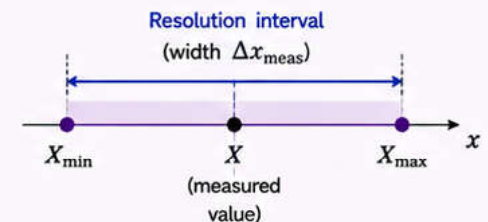
$$\Delta x_{\text{meas}} \geq \frac{\Delta x_{\text{inst}}}{N}$$

Averaging over N periods improves the achievable resolution.

$$\text{Example: } N = 40 \rightarrow \Delta x_{\text{meas}} \geq 0.5 \text{ s} / 40 = 0.0125 \text{ s}$$

Resolution Defines an Interval

If a measurement is performed with a measurement resolution Δx_{meas} , the true value X can be anywhere within an interval of width Δx_{meas} .



$$X_{\min} \leq X \leq X_{\max}$$

Assuming all values in the interval are equally likely:

Uniform (constant) probability



Key result

If all values in the resolution interval are equally likely, the maximum possible deviation from the measured value is:

Maximum uncertainty due to resolution

$$u_{\text{res,max}} = \pm \frac{\Delta x_{\text{meas}}}{2}$$

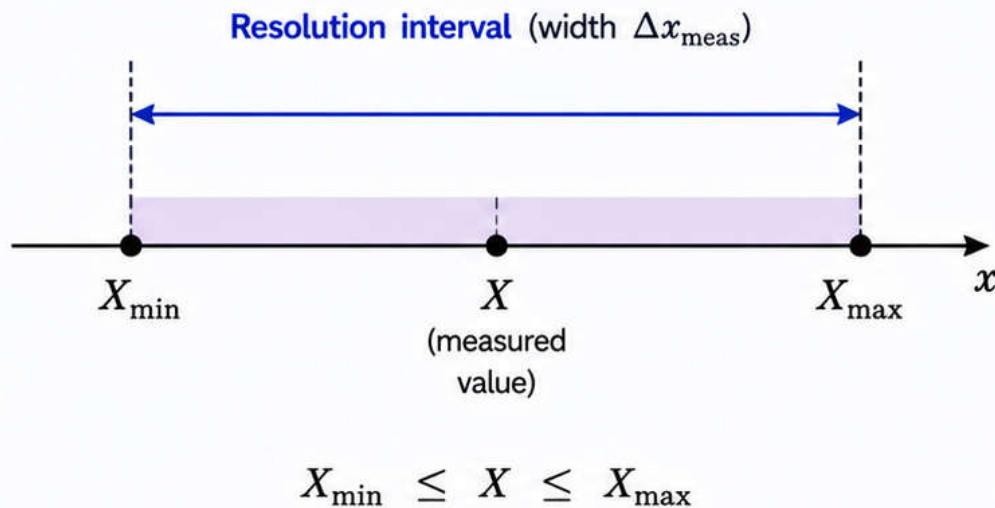
We will later see that the standard uncertainty associated with a rectangular distribution is $u_{\text{res}} = \frac{\Delta x_{\text{meas}}}{2\sqrt{3}}$.



Resolution Defines an Interval



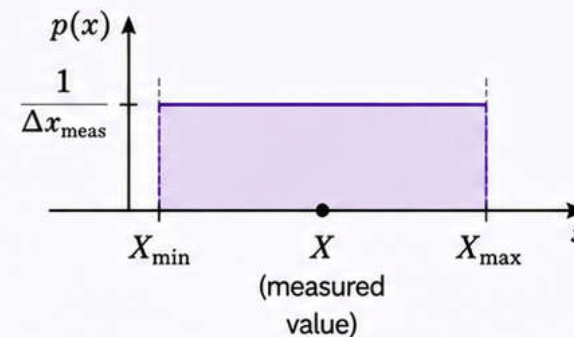
If a measurement is performed with a measurement resolution Δx_{meas} , the true value X can be **anywhere within an interval of width Δx_{meas}** .



What do we know?

- We know the measured value X (e.g., the reading).
- We know the resolution Δx_{meas} .
- We **do not know** where exactly within the interval the true value lies.

Assuming all values in the interval are equally likely:



Uniform (constant) probability within the interval $[X_{\min}, X_{\max}]$.

Outside the interval: $p(x) = 0$.



Important: The resolution does not tell us the true value. It only tells us the limits within which the true value must lie.



The **uncertainty due to resolution** comes from this interval.



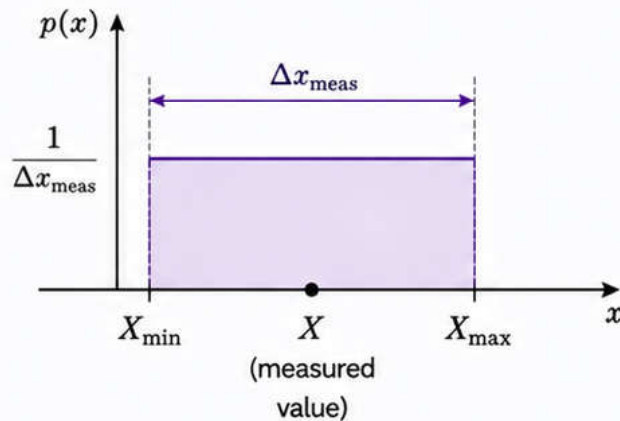
Uncertainty Due to Resolution



Since the true value can be **anywhere** within the resolution interval of width Δx_{meas} and all values in this interval are equally likely, the **uncertainty due to resolution** is:

$$u_{\text{res}} = \frac{\Delta x_{\text{meas}}}{2\sqrt{3}}$$

Derivation (assuming uniform distribution)



Mean (expected value):

$$\bar{x} = \int_{X_{\min}}^{X_{\max}} x p(x) dx = X$$

Variance:

$$u^2 = \int_{X_{\min}}^{X_{\max}} (x - \bar{x})^2 p(x) dx = \frac{\Delta x_{\text{meas}}^2}{12}$$



Standard uncertainty due to resolution:

$$u_{\text{res}} = \sqrt{u^2} = \frac{\Delta x_{\text{meas}}}{\sqrt{12}} = \frac{\Delta x_{\text{meas}}}{2\sqrt{3}}$$



Key ideas

- The **resolution** limits the possible values to an interval.
- All values in this interval are equally likely.
- The standard uncertainty is one half of the interval divided by $\sqrt{3}$.
- This uncertainty is of **Type B** (evaluated from other information).

Example: Digital stopwatch



Display resolution: 0.01 s

$$\Delta x_{\text{meas}} = 0.01 \text{ s}$$



$$u_{\text{res}} = \frac{0.01}{2\sqrt{3}} = 0.00289 \text{ s}$$

We cannot state the time more precisely than this due to the display resolution.



Summary

The resolution of a measuring instrument does not give the true value, but limits it to an interval. This leads to an uncertainty of

$$u_{\text{res}} = \frac{\Delta x_{\text{meas}}}{2\sqrt{3}}$$

Remember

This formula is valid only when the distribution is uniform over the resolution interval.



Random Fluctuations (Random Error)



Random fluctuations are unpredictable variations in repeated measurements caused by many small, uncontrollable influences.



What are they?

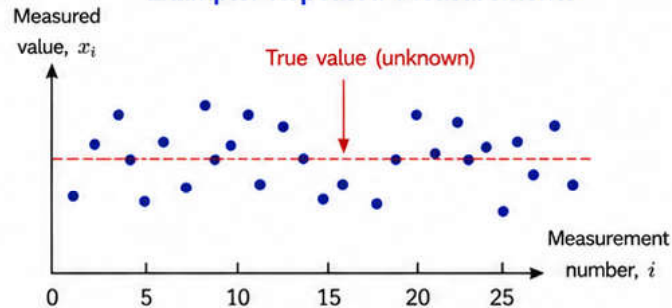
- Cause measured values to scatter around the true value.
- Have zero mean over many measurements.
- Their effect can be reduced by averaging.



How do they affect measurements?

- They limit how precisely we can know the true value.
- The scatter of results quantifies the uncertainty due to random fluctuations.
- More measurements $\rightarrow$ better estimate of the mean.

Example: Repeated Measurements



Individual results vary randomly about the true value.
The mean of many results is our best estimate.



Key ideas

- Random error is **not** a bias. It does not shift the mean.
- It only increases the uncertainty of the mean.
- Its reduction follows the $1/\sqrt{N}$ law.



Effect of averaging

The standard uncertainty of the mean decreases with the number of measurements N .

$$u_{\text{rand}}(\bar{x}) = \frac{s}{\sqrt{N}}$$

where s is the sample standard deviation.

Typical sources of random fluctuations



Small temperature variations



Electrical noise and interference



Mechanical vibrations and disturbances



Timing jitter and resolution



Many other small, uncontrollable factors



Example

A length is measured 25 times with $s = 0.006$ mm. Standard uncertainty of the mean:

$$u_{\text{rand}}(\bar{x}) = \frac{0.006}{\sqrt{25}} = 0.0012 \text{ mm}$$



Random fluctuations cannot be eliminated, but their impact can be **quantified** and **reduced by averaging**.



Two Experimental Situations

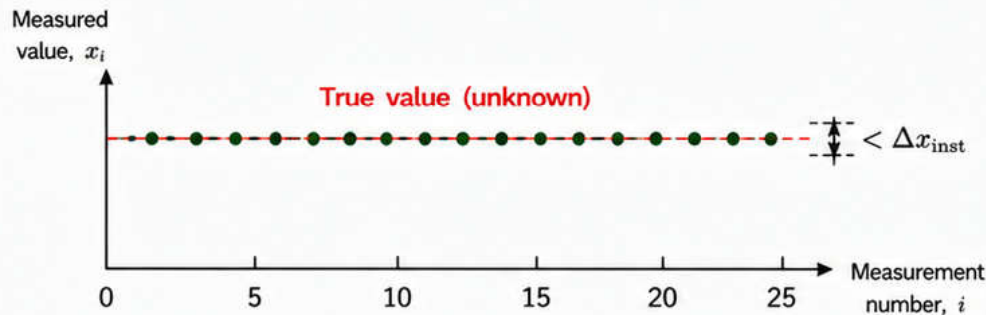


Random fluctuations in repeated measurements can be either very small or large compared to the instrument resolution.

A

Fluctuations smaller than resolution

All measurements show the same value.

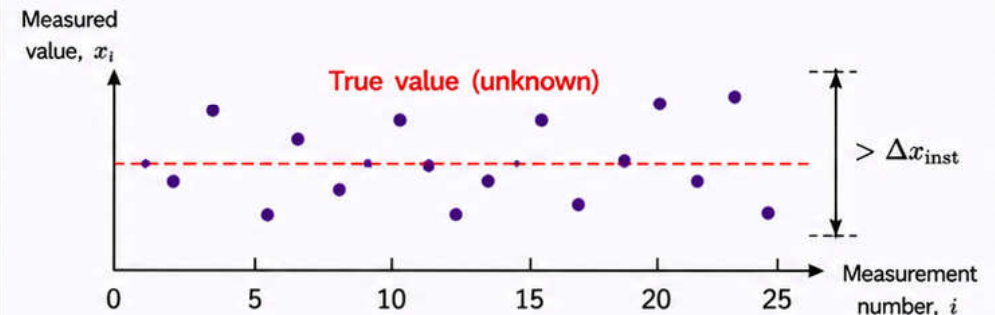


- The scatter is smaller than the instrument resolution Δx_{inst} .
- All readings are indistinguishable.
- The instrument **cannot reveal** the small fluctuations.

B

Fluctuations larger than resolution

The measured values show visible scatter.



- The scatter is larger than the instrument resolution Δx_{inst} .
- The readings vary from one measurement to another.
- The instrument **can reveal** the fluctuations.



Important: Which situation occurs depends on the size of the random fluctuations relative to the instrument resolution.

We will now see how improving resolution can change what we observe.



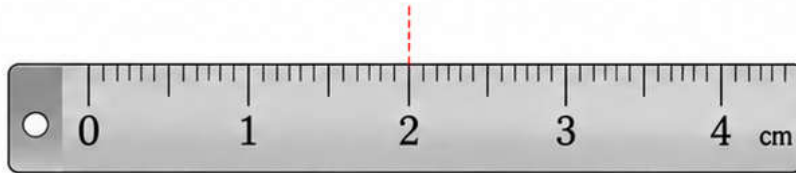
Better Resolution Reveals More Variability



The same quantity is measured with instruments of different resolution.
Higher resolution can **reveal** random fluctuations that were previously **hidden**.

A Ruler

$\Delta x_{\text{inst}} = 0.5 \text{ mm}$ → apparently identical values

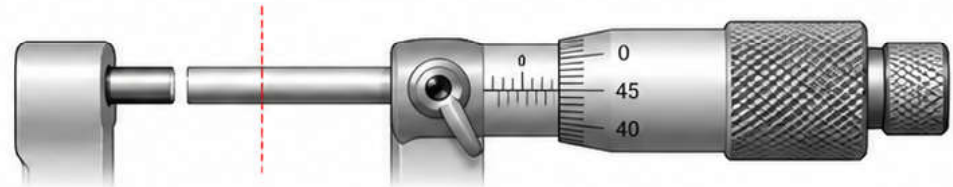


Repeated measurements	1	2	3	...	N
$x_i \text{ (mm)}$	2.5	2.5	2.5	...	2.5

- All readings are the same (within the resolution).
- Fluctuations are **smaller** than the resolution.

B Micrometer

$\Delta x_{\text{inst}} = 0.005 \text{ mm}$ → visible fluctuations



Repeated measurements	1	2	3	...	N
$x_i \text{ (mm)}$	2.503	2.497	2.501	...	2.498

- Readings vary from one measurement to another.
- Fluctuations are **larger** than the resolution and are **revealed**.



Key takeaway: Improving resolution can reveal previously hidden random variability.

What looked “perfectly constant” at low resolution may actually be fluctuating.



Repeated Measurements Produce a Sample



Because of random fluctuations, repeating the same measurement gives a set of slightly different results. This set is called a **sample**.



The sample

$$x_1, x_2, x_3, \dots, x_N$$

where

- x_i is the i -th measurement result
- N is the number of repeated measurements

Notation

$N =$ **sample size**



What does the sample represent?

- Each x_i is one possible outcome of the measurement.
- The set of all x_i contains information about the true value and the random fluctuations.
- We do not know the true value, but we can describe the sample statistically.

$x_1, x_2, \dots, x_N$



sample



statistical
description



Key idea: We cannot predict each individual measurement, but we can analyze the sample as a whole.

Repeated measurements $\rightarrow$ sample $\rightarrow$ statistical description



A sample is the **foundation** for estimating the true value and its uncertainty.



From Measurements to a Histogram



A sample of repeated measurements can be organized into a **histogram**, which shows how the values are distributed.

1 Raw measurements

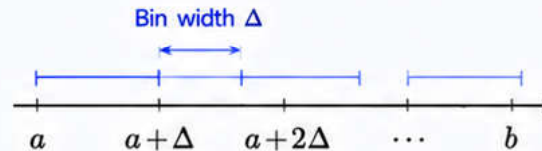
Start with the list of N repeated measurements.

$$x_1, x_2, x_3, \dots, x_N$$

These are individual outcomes that fluctuate from one measurement to another.

2 Bins (classes)

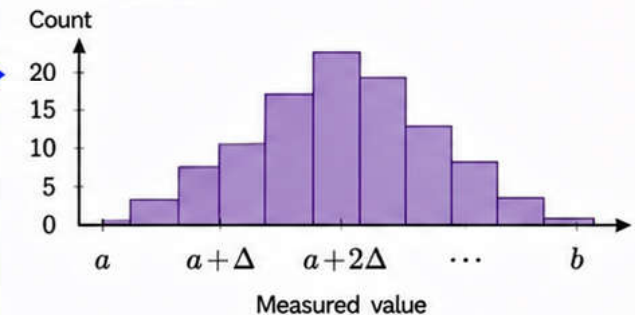
Divide the range of values into intervals (bins or classes).



Count how many measurements fall into each bin.

3 Histogram

Plot the counts (or relative frequencies) as a bar chart.



Key takeaway:

The histogram contains information about both **location** and **spread**.

- The center of the histogram indicates the **location** (typical value).
- The width of the histogram indicates the **spread** (variability).



Important: The histogram is a bridge from raw measurements to statistical analysis. It allows us to visualize and quantify random fluctuations.





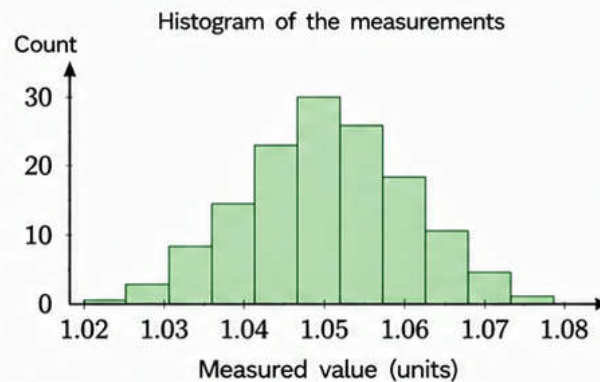
How Can We Summarize a Distribution?



A histogram (or frequency distribution) contains a lot of information, but it is often convenient to summarize the distribution using a few **quantitative numbers**.

Example: Repeated measurements

Measurement i	Value x_i
1	1.045
2	1.052
3	1.048
$\vdots$	$\vdots$
N	1.051



What summary numbers do we want?

We want numbers that describe:

- the central or typical value of the distribution (**location**)
- the amount of spread or scatter of the values (**variability**)

Key idea:



Two key numbers are especially important:

- 1 A measure of location → the **sample mean** $\bar{x}$
- 2 A measure of spread → the **standard deviation** s



Goal of this block

We will define and understand the sample mean, variance, and standard deviation, and learn how to calculate them.

Why are these useful?

- They allow us to summarize a distribution succinctly.
- They are the foundation for estimating the true value and its uncertainty.



Takeaway: A distribution can be summarized by its typical value (mean) and its spread (standard deviation).



Sample Mean



The **sample mean** (or arithmetic mean) is the average of all the measurements. It is our best estimate of the typical or central value (**location**) of the distribution.



Definition

For a set of N measurements

$x_1, x_2, \dots, x_N$, the sample **mean** $\bar{x}$ is

$$\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i = \frac{x_1 + x_2 + \dots + x_N}{N}$$

where

- x_i is the i -th measurement
- N is the number of measurements (sample size)
- $\bar{x}$ is the sample mean



Example

Suppose we have the following 5 measurements:

1.045, 1.052, 1.048, 1.051, 1.050

The sample mean is

$$\bar{x} = \frac{1.045 + 1.052 + 1.048 + 1.051 + 1.050}{5}$$

$$\bar{x} = 1.0492$$



The sample mean represents the **center** of the data.



Key takeaway: The sample mean $\bar{x}$ provides a single number that represents the **typical (central) value** of the measurements.

Location measure $\rightarrow$ sample mean $\bar{x}$



Deviations from the Mean



To quantify the spread of the data, we look at how far each measurement is from the sample mean. These differences are called **deviations**.



Definition

The deviation of the i -th measurement from the sample mean $\bar{x}$ is

$$d_i = x_i - \bar{x}$$

where

- x_i is the i -th measurement
- $\bar{x}$ is the sample mean
- d_i is the deviation of x_i



Example

Using the measurements from the previous slide:

$$\bar{x} = 1.0492$$

i	x_i	$d_i = x_i - \bar{x}$	Interpretation
1	1.045	$1.045 - 1.0492 = -0.0042$	below mean
2	1.052	$1.052 - 1.0492 = +0.0028$	above mean
3	1.048	$1.048 - 1.0492 = -0.0012$	below mean
4	1.051	$1.051 - 1.0492 = +0.0018$	above mean
5	1.050	$1.050 - 1.0492 = +0.0008$	above mean

$$\text{Sum of deviations: } \sum_{i=1}^5 d_i = 0 \text{ (up to rounding)}$$



Interpretation

- Negative deviations are below the mean.
- Positive deviations are above the mean.
- Deviations tell us the direction and magnitude of each measurement's departure from the mean.
- The deviations always sum to zero (up to rounding).

Deviations are the building blocks for measuring the spread.



Key takeaway: Deviations measure how far each data point is from the sample mean.

Deviation = measurement – mean → basis for spread



Variance



Variance is a measure of the **spread** of the data.
It is the average of the **squared deviations** from the sample mean.



Definition

For a set of N measurements $x_1, x_2, \dots, x_N$ with sample mean $\bar{x}$, the variance s^2 is

$$s^2 = \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2 = \frac{1}{N-1} \sum_{i=1}^N d_i^2$$

where

- x_i is the i -th measurement
- $\bar{x}$ is the sample mean
- $d_i = x_i - \bar{x}$ is the deviation of x_i
- N is the number of measurements (sample size)
- The denominator $(N-1)$ is called **Bessel's correction**.
It makes s^2 an unbiased estimate of the population variance.



Example

Using the deviations from the previous slide:

i	x_i	$d_i = x_i - \bar{x}$	d_i^2
1	1.045	-0.0042	0.00001764
2	1.052	+0.0028	0.00000784
3	1.048	-0.0012	0.00000144
4	1.051	+0.0018	0.00000324
5	1.050	+0.0008	0.00000064

Sum of squared deviations: $\sum d_i^2 = 0.00003080$

Variance:

$$s^2 = \frac{0.00003080}{5-1} = 0.00000770$$



Interpretation

- Squaring the deviations makes all contributions positive and gives more weight to larger deviations.
- Variance has units of (measured value)².
- A larger variance means the data are more spread out.

Variance quantifies the spread of the distribution.



Key takeaway: Variance is the average of the squared deviations and is a fundamental measure of spread.

Spread measure $\rightarrow$ variance s^2



Sample Standard Deviation



The **sample standard deviation** is the square root of the variance. It has the **same units** as the measurements and is easier to interpret.



Definition

The sample standard deviation s is the positive square root of the variance s^2 :

$$s = \sqrt{s^2} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2}$$

where

- x_i is the i -th measurement
- $\bar{x}$ is the sample mean
- N is the number of measurements (sample size)
- The denominator $(N - 1)$ is **Bessel's correction**. It makes s an unbiased estimate of the population standard deviation.



Example

From the previous slide:

$$s^2 = 0.00000770$$

Therefore, the sample standard deviation is

$$s = \sqrt{s^2} = \sqrt{0.00000770}$$

$$s = 0.00277$$



The standard deviation tells us the **typical size** of the deviations from the mean.



Interpretation

- About 68% of the measurements are expected to lie within $\pm 1s$ of the mean (for roughly symmetric distributions).
- About 95% lie within $\pm 2s$.
- About 99.7% lie within $\pm 3s$.
- The standard deviation is widely used because it has the same units as the data and is easy to understand.

Standard deviation provides a direct, intuitive measure of the typical spread of the data.



Key takeaway: The sample standard deviation s is the typical (root-mean-square) deviation of the measurements from the mean.

Spread measure $\rightarrow$ standard deviation s



An Alternative Formula for the Variance



Computing the variance using deviations requires calculating the mean first.

There is an **equivalent formula** that uses only the measurements themselves, which is often more convenient.



Derivation

Starting from the definition of the variance:

$$s^2 = \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2$$

Expand and simplify:

$$\begin{aligned} \sum (x_i - \bar{x})^2 &= \sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2) \\ &= \sum x_i^2 - 2\bar{x} \sum x_i + \sum \bar{x}^2 \\ &= \sum x_i^2 - 2\bar{x} (N\bar{x}) + N\bar{x}^2 \\ &= \sum x_i^2 - 2N\bar{x}^2 + N\bar{x}^2 \\ &= \sum x_i^2 - N\bar{x}^2 \end{aligned}$$

Therefore,

$$s^2 = \frac{\sum_{i=1}^N x_i^2 - N\bar{x}^2}{N-1}$$

This formula is algebraically equivalent to the original one.



Alternative Formula

The variance can be calculated as

$$s^2 = \frac{\sum_{i=1}^N x_i^2 - N\bar{x}^2}{N-1}$$

where

- x_i are the measurements
- $\bar{x}$ is the sample mean
- N is the number of measurements



Either formula gives the same result (up to rounding).
Use whichever is more convenient.



When is it useful?

The alternative formula is convenient when:

- you already have $\bar{x}$ and $\sum x_i^2$ from other calculations,
- or when using a calculator/spreadsheet, which can compute it efficiently.

Numerical stability

If the data have very large values with small spread, the deviation formula may be more numerically stable.

Rule of thumb: Try both methods and compare the results.



Key takeaway: The variance can be computed using only the measurements:

$$s^2 = \frac{\sum_{i=1}^N x_i^2 - N\bar{x}^2}{N-1}$$



Same result,
often more convenient!



What Happens When N Increases?



As the number of measurements N increases, the histogram of the data becomes **smoother** and begins to reveal the underlying pattern of the distribution.



Key Idea

With a small number of measurements, the histogram looks irregular and noisy. As N increases, random fluctuations average out, and a stable shape emerges.



Larger $N \rightarrow$ more reliable histogram
 $\rightarrow$ clearer picture of the distribution

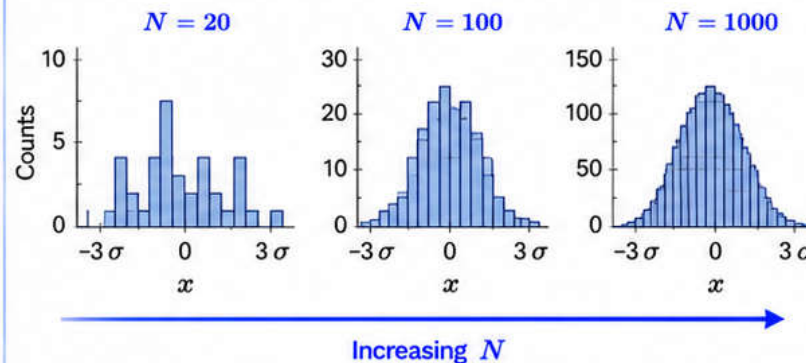
Example Setup

- Measuring a quantity x (e.g., period of a pendulum)
- True distribution is approximately normal
- We build histograms with different sample sizes N using the same bin width Δx



Effect of Increasing N

Example histograms for the same measurement with increasing sample size N .



As N increases, the histogram becomes smoother, more symmetric, and better approximates the true distribution.



Why This Matters

- Random errors average out as N grows.
- The histogram approaches a stable shape that reflects the underlying probability distribution.
- This motivates us to think of the histogram as an approximation to a continuous distribution.



Takeaway

Increasing N turns a noisy histogram into a smooth, reliable representation of the distribution.



Key takeaway: As the number of measurements N increases, the histogram becomes smoother and reveals the underlying distribution more clearly.

Small $N \rightarrow$ noisy histogram $\rightarrow$ Large $N \rightarrow$ smooth histogram



The Limiting Histogram



As $N \rightarrow \infty$ and the bin width $\Delta x \rightarrow 0$, the histogram approaches a smooth curve. This is the **limiting histogram** and represents the underlying probability distribution.



Definition

The limiting histogram is the result of taking more and more measurements and using smaller and smaller bins.

$$\lim_{\substack{N \rightarrow \infty \\ \Delta x \rightarrow 0}} \text{histogram} = \text{smooth curve}$$

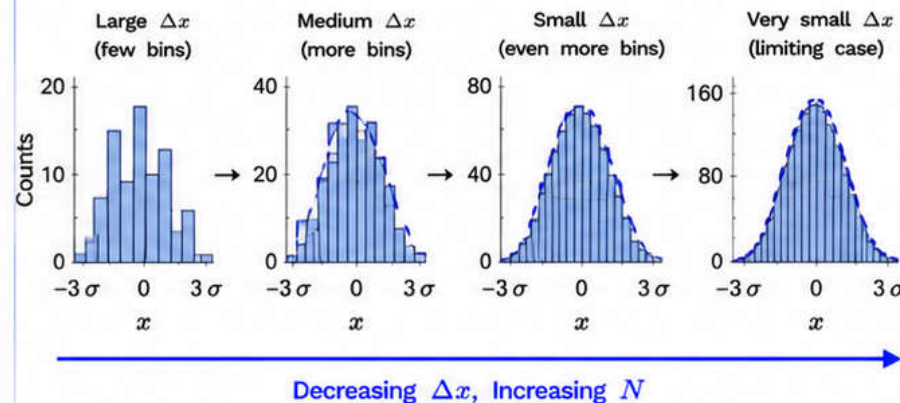
In this limit:

- Random fluctuations disappear.
- The top of each bar approaches a definite height.
- The histogram tends to a **continuous function** $f(x)$.



Approach to the Limit

Histogram of the same data with decreasing bin width.



In the limiting case, the histogram becomes a smooth curve. This curve is the **underlying probability distribution** $f(x)$.



Properties of the Limiting Histogram

- The area under the curve equals 1

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

- $f(x) \geq 0$ for all x
- $f(x)$ gives the **probability density**:

$$P(a \leq x \leq b) = \int_a^b f(x) dx$$

The limiting histogram is the bridge from discrete data to a continuous probability distribution.



Key takeaway: The limiting histogram is what the histogram becomes when we take infinitely many measurements and use infinitely small bins. It is a smooth curve that represents the true probability distribution.

More data + smaller bins $\Rightarrow$ limiting histogram
= underlying distribution $f(x)$



From Discrete Histogram to Continuous Distribution



The histogram uses discrete bins and counts. The continuous distribution replaces the bars by a **smooth probability density function** that describes the data for any value of x .



1. From Counts to Density

A histogram shows counts in each bin. To obtain a continuous distribution, we convert counts to **density (area)**.

$$\text{density (height)} = \frac{\text{counts in bin}}{\text{total count} \times \text{bin width}}$$

- This ensures that the total area under the curve is 1.
- The height now depends on bin width.



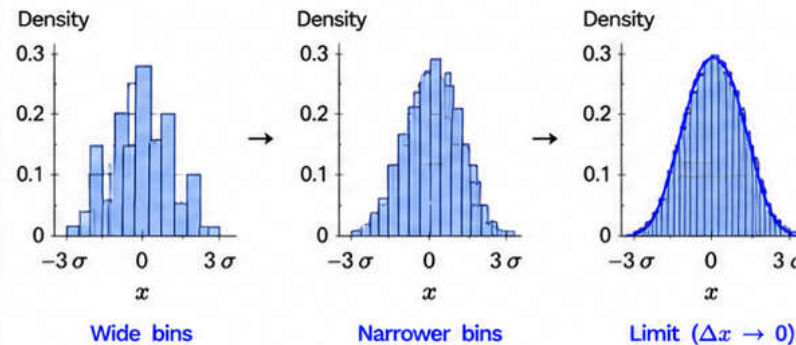
Why density?

Because probability corresponds to area, **not to height**.



2. Building the Continuous Curve

As the bin width Δx becomes smaller, the tops of the bars trace out a smooth curve.

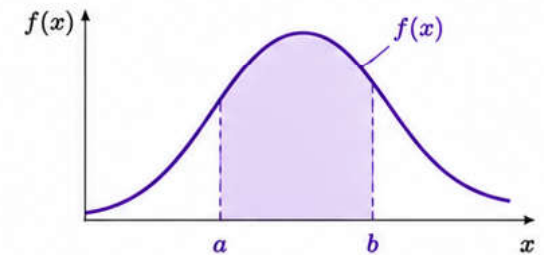


In the limit of infinitely small bins, the histogram becomes the smooth probability density function $f(x)$.



3. Reading Probabilities

In the continuous picture, probabilities are areas under the curve.



$$P(a \leq x \leq b) = \int_a^b f(x) dx$$

The total area under $f(x)$ over all x equals 1.



Key takeaway: A histogram with smaller and smaller bins approaches a smooth curve. This curve, the probability density function $f(x)$, describes the distribution for any x and allows probabilities to be found as areas under the curve.

Counts in bins $\Rightarrow$ Density in bins $\Rightarrow$ Smooth density function $f(x)$



The Normal Distribution



In many measurement situations, the limiting distribution is well described by the **normal (Gaussian) distribution**. It is completely determined by its mean μ and standard deviation σ .



1. Definition

The normal distribution with mean μ and standard deviation σ has probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

for $-\infty < x < \infty$, $\sigma > 0$.

Properties:

- Bell-shaped and symmetric about μ
- Total area under the curve equals 1
- Mean = median = mode = μ
- Completely determined by μ and σ



Standard Normal Distribution

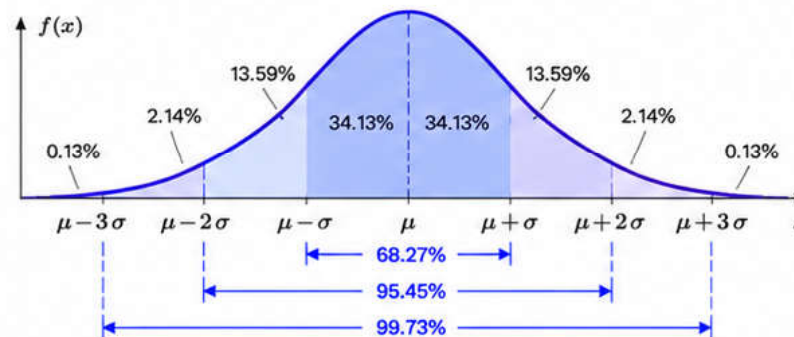
If $\mu = 0$ and $\sigma = 1$, we get the standard normal distribution $Z \sim N(0,1)$ with

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$



2. Shape and Key Features

The normal distribution is symmetric and bell-shaped.



Empirical Rule (68–95–99.7 Rule)

For a normal distribution:

- 68.27% of values lie within $\mu \pm 1\sigma$
- 95.45% lie within $\mu \pm 2\sigma$
- 99.73% lie within $\mu \pm 3\sigma$

This rule holds approximately for many measurements in practice.



3. Why It Appears So Often

- Many small, independent random effects add together.
- By the **Central Limit Theorem**, their sum (or average) is approximately normal.
- Hence, measurement errors and many physical quantities follow a normal distribution.

Examples

- Repeated measurements (random errors)
- Electronic noise
- Heights, weights, and many natural variations
- Results from the central limit theorem



Key Point

Once you know μ and σ , you know the entire distribution and can compute any probability.



Key takeaway: The normal distribution is the most important continuous distribution in measurement. It is symmetric, bell-shaped, and fully described by its mean μ and standard deviation σ .

Normal distribution: $N(\mu, \sigma^2)$

Special case: $N(0, 1)$ (standard normal)



38. Random Uncertainty of a Direct Measurement



Every direct measurement has random uncertainty due to unpredictable effects that cause the measured values to scatter around the true value.



1. What Causes Random Uncertainty?

Many small, uncontrollable effects contribute:

- Electronic noise
- Environmental fluctuations (temperature, vibration, ...)
- Imperfect instrument response
- Human reading variations
- Other random disturbances



These effects are unpredictable in detail, but their combined effect is random.



2. Statistical Model

The measured values scatter randomly around the true value.

Assume a normal (Gaussian) distribution:

$$y_i \sim \mathcal{N}(\mu, \sigma^2)$$

where

- μ : true value
- σ : standard deviation (random uncertainty)
- y_i : i -th measurement



3. Estimating Random Uncertainty

From N repeated measurements $y_1, y_2, \dots, y_N$:

Best estimate of the true value:

$$\bar{y} = \frac{1}{N} \sum_{i=1}^N y_i$$

Experimental standard deviation:

$$s = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (y_i - \bar{y})^2}$$

Standard uncertainty of the mean:

$$u(\bar{y}) = \frac{s}{\sqrt{N}}$$



4. Key Points

- Random uncertainty cannot be eliminated, but it can be reduced.
- More repeated measurements (larger N) reduce the uncertainty of the mean.
- Report measurements as:

$$\bar{y} \pm u(\bar{y}) \quad (k = 1)$$

or

$$\bar{y} \pm k u(\bar{y})$$

where k is the coverage factor (e.g., $k = 2$ for $\sim 95\%$ confidence).



Key takeaway: Random uncertainty in a direct measurement comes from many small, unpredictable effects. It is characterized by a standard deviation and decreases as more measurements are averaged.

More measurements (larger N) $\Rightarrow$ smaller uncertainty of the mean



Random uncertainty is a fundamental concept in measurement and forms the basis for least squares fitting and uncertainty analysis.



Summary: From Measurement to Meaning



1. Measurement Basics

- Measurement quantifies physical quantities by comparison.
- Understand the measurement process and the instruments used.
- Every measurement has uncertainty.

Measure carefully. Know what you are measuring.



2. Data and Distributions

- Repeated measurements produce data that scatter.
- Histograms reveal the underlying distribution.
- In many cases, the distribution is well described by a normal curve.

Data contain information.
Look for patterns.



3. Uncertainty

- Uncertainty has random and systematic components.
- Random uncertainty is characterized by standard deviation.
- More data reduce random uncertainty.

Quantify uncertainty.
It is part of the result.



4. Modeling and Fitting

- Models describe relationships between variables.
- Least squares fitting finds the best model parameters.
- Goodness of fit must be checked.

A good model + good data
= reliable results.



5. From Theory to Practice

- Combine theory, measurement, and statistics.
- Report results with uncertainty.
- Make informed conclusions and decisions.

Turn measurements
into knowledge.



The big picture: Careful measurement, thoughtful analysis, and honest reporting transform raw data into meaningful, trustworthy knowledge.