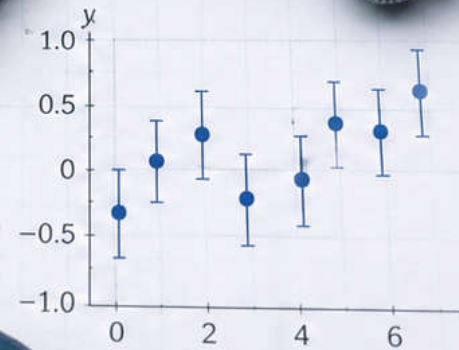
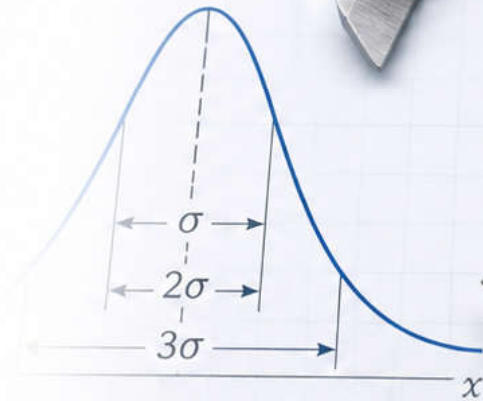


Propagation of Measurement Uncertainty I

From directly measured quantities to calculated results



How uncertainties in measured quantities affect indirectly calculated results



$$Q = f(X_1, X_2, \dots, X_n) \pm \delta Q$$

From the previous lecture

Probability distributions → measurement uncertainty



Binomial distribution

Number of successes in n independent trials with success probability p .



Poisson distribution

Number of random events in a fixed interval with known average rate λ .



Normal distribution

Continuous distribution; central role in measurement and natural processes.



Central limit theorem

The sum/average of many independent contributions is (approximately) normally distributed.

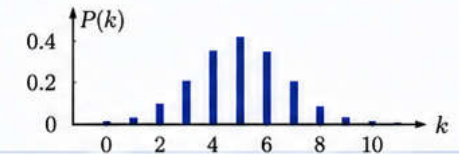


Cauchy-Lorentz distribution

Heavy-tailed continuous distribution; mean and variance are not defined.

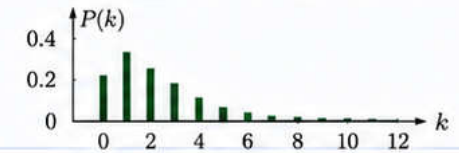
Binomial

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



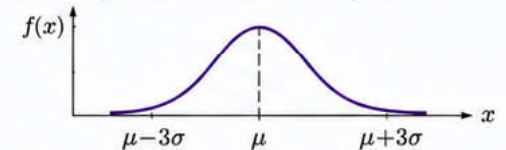
Poisson

$$P(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$



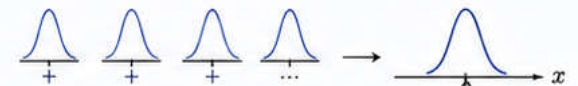
Normal

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



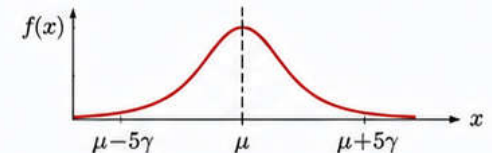
Central limit theorem

Sums / averages of many independent variables → Normal



Cauchy-Lorentz

$$f(x) = \frac{1}{\pi} \frac{\gamma}{(x-\mu)^2 + \gamma^2}$$



Now we ask:

What happens to uncertainty when measured quantities are used in a calculation?

Direct vs. Indirect Measurement

We distinguish between quantities measured *directly* by an instrument and quantities obtained *indirectly* through a calculation.

DIRECT

A physical quantity is measured directly by an instrument.



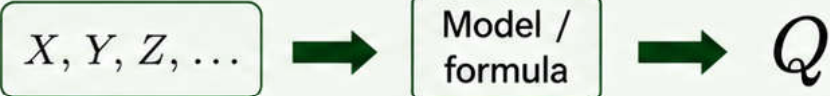
$$X = X_0 \pm \delta X$$

Example:

Length measured with a caliper: $l = (25.30 \pm 0.02) \text{ mm}$

INDIRECT

A quantity is obtained by a calculation from measured quantities.



$$Q = f(X, Y, Z, \dots)$$

$$Q = Q_0 \pm \delta Q$$

Example:

Pendulum: $g = 4\pi^2 \frac{l}{T^2}$ from measured l, T we calculate g .



Key message:

Direct measurements give $X = X_0 \pm \delta X$. Indirect measurements give $Q = Q_0 \pm \delta Q$. How do we find δQ ?

Example: Measuring g with a Pendulum

Physical situation

- The acceleration due to gravity g can be determined with a simple pendulum.

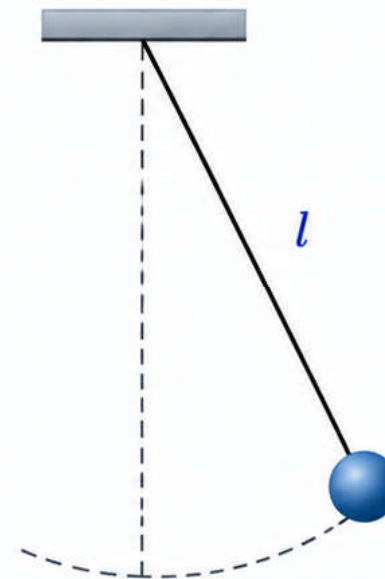
Directly measured quantities

- Length of the pendulum: l $\rightarrow l = l_0 \pm \delta l$
- Period of oscillation: T $\rightarrow T = T_0 \pm \delta T$

Calculated (indirectly obtained) quantity

$$g = 4\pi^2 \frac{l}{T^2} \quad \text{From measurements of } l \text{ and } T \text{ we calculate } g.$$

Simple pendulum



$$T = 2\pi\sqrt{\frac{l}{g}}$$

l – length
[m]

T – period
[s]

g – acceleration
due to gravity
[m/s²]



Question:

If l and T have uncertainties, how uncertain is the calculated g ?

Example: Propagation for a function of one variable

Pendulum model

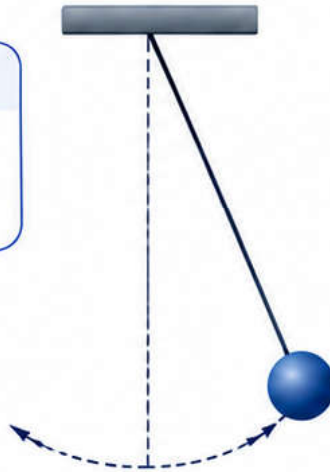
$$g = 4\pi^2 \frac{l}{T^2} \quad \left| \quad \begin{array}{l} \text{Indirectly measured quantity:} \\ g = f(l, T) \end{array} \right.$$

Measured quantities

$$\begin{array}{l} l = (1.000 \pm 0.005) \text{ m} \\ T = (2.006 \pm 0.010) \text{ s} \end{array} \quad \left(\begin{array}{l} \text{independent measurements} \end{array} \right.)$$

Partial derivatives

$$\frac{\partial g}{\partial l} = \frac{4\pi^2}{T^2}, \quad \frac{\partial g}{\partial T} = -\frac{8\pi^2 l}{T^3}$$



Propagation of uncertainty

$$\delta g = \sqrt{\left(\frac{\partial g}{\partial l} \delta l \right)^2 + \left(\frac{\partial g}{\partial T} \delta T \right)^2}$$

Calculation

$$\frac{\partial g}{\partial l} = \frac{4\pi^2}{T^2} = 1.976 \text{ m/s}^2 \text{ m}^{-1}$$

$$\frac{\partial g}{\partial T} = -\frac{8\pi^2 l}{T^3} = -1.970 \text{ m/s}^2 \text{ s}^{-1}$$

$$\delta g = \sqrt{(1.976 \cdot 0.005)^2 + (1.970 \cdot 0.010)^2} = 0.022 \text{ m/s}^2$$

$$\text{Result: } g = 9.81 \pm 0.02 \text{ m/s}^2$$



Key message:

For a function of several independent variables, uncertainties propagate via the **partial derivatives** (sensitivity coefficients), contributions **combine in quadrature**.

What is Uncertainty Propagation?

In many experiments, the quantity of interest is **not measured** directly, but is calculated from several measured quantities. How do their **uncertainties** affect the result of the calculation?

Definition

Uncertainty propagation describes how uncertainties in input quantities produce uncertainty in a calculated output quantity.

Formal statement

Directly measured quantities: $X, Y, Z, \dots$

Calculated (indirect) quantity: $Q = f(X, Y, Z, \dots)$



$$Q = Q_0 \pm \delta Q$$

From inputs to result

Measured
inputs

$$\begin{aligned} X &= X_0 \pm \delta X \\ Y &= Y_0 \pm \delta Y \\ Z &= Z_0 \pm \delta Z \\ &\vdots \end{aligned}$$

Measurement
model

$$Q = f(X, Y, Z, \dots)$$

Calculated
result

$$Q = Q_0 \pm \delta Q$$

Goal of this lecture:

- Understand how uncertainties combine in calculations.
- Learn practical rules for linear and nonlinear relationships.
- Apply these rules to real measurement examples.



Central question:

Given the uncertainties of the measured quantities, how do we determine δQ ?

6. Independent Measurements

DEFINITION

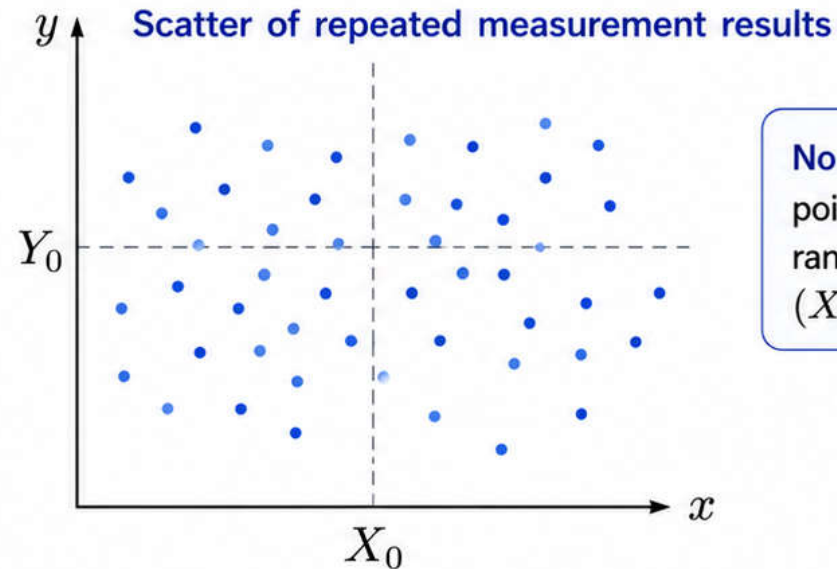
Two measured quantities X and Y are **statistically independent** if their deviations from their means are independent.

$x - X_0$ and $y - Y_0$ do not systematically vary together.

INTUITIVELY

When we repeat the measurements many times, the random deviations of X from X_0 should not influence the random deviations of Y from Y_0 , and vice versa.

- Knowing one deviation does not help to predict the other.



No correlation:
points are scattered
randomly around
(X_0, Y_0).

7. Independence is About Fluctuations

Any measurement result can be written as:

$$X = X_0 + \Delta X \quad \text{and} \quad Y = Y_0 + \Delta Y$$

where:

X_0, Y_0 are the (true) mean values,

$\Delta X, \Delta Y$ are the random deviations from the means.



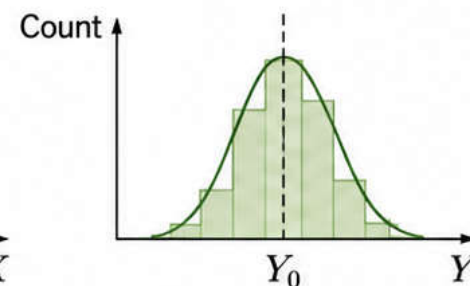
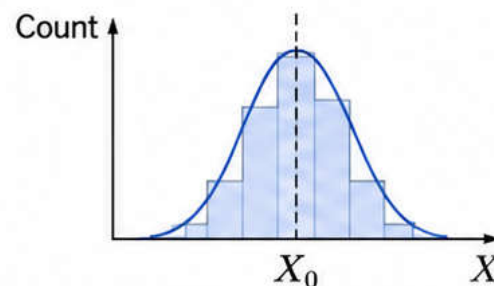
Key idea:

Statistical independence concerns the fluctuations ΔX and ΔY , not simply whether X and Y are related by a physical law.

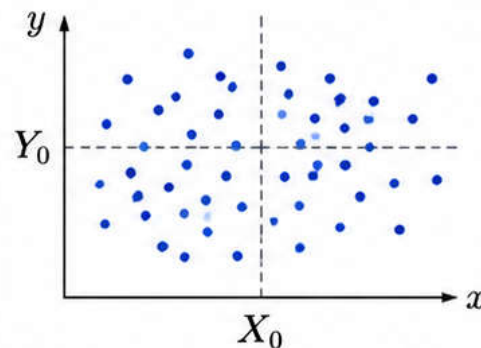
In summary:

- X and Y may be physically related.
- They are statistically independent if their deviations ΔX and ΔY do not systematically vary together.
- The fluctuations in one measurement give no information about the fluctuations in the other.

Visualizing repeated measurements

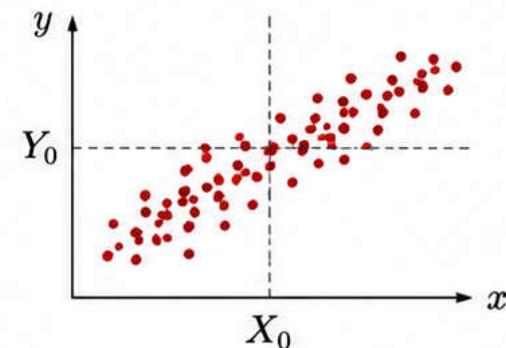


Independent (no correlation)



Points are scattered randomly around (X_0, Y_0) .

Dependent (correlated)



Knowing x (or y) gives information about y (or x).



Statistical independence is about the behavior of random deviations, not about physical relations.

8. Physical Relation $\neq$ Statistical Dependence



Important point:

Two quantities can be related by a **physical** law (functional dependence), yet their measurement deviations can still be **statistically independent**.

Example: simple pendulum

The period of a simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

T – period [s]
 l – length [m]
 g – acceleration due to gravity [m/s^2]

- Thus, T and l are **physically** (functionally) related.

BUT:

Random deviations in measuring T (e.g., reaction time, counting error) may be **independent** of random deviations in measuring l (e.g., reading the scale, instrument resolution).

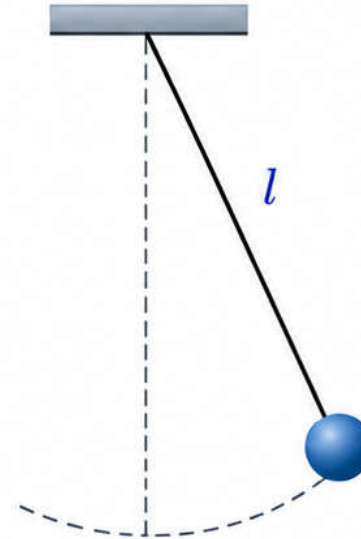


Functional dependence does not imply statistical dependence.



Take-home message:

Physical (functional) relations connect the **mean values**, not the **random deviations**.



Measuring the period T

- Start/stop timing
- Human reaction time
- Counting cycles
- Electronic timer resolution

⇒ **Random deviations in T**

Measuring the length l

- Reading the scale
- Parallax error
- Instrument resolution
- Thermal expansion, etc.

⇒ **Random deviations in l**



Conclusion:

Deviations in measuring T do not influence deviations in measuring l , and vice versa → they can be **statistically independent**.

9. Example of Non-Independent Measurements



Situation

We measure the sides of a rectangle using the same tape measure that has a **systematic bias**.

What happens?

The tape measure is either too long or too short by (approximately) the same amount for all measurements.



$$\begin{array}{l} \text{True length} \\ \text{Reading} \\ = \text{True length} + b \end{array}$$

Let the systematic bias be b .

Measured values: where ϵ_x, ϵ_y are (small) random fluctuations.
 $x = X_0 + b + \epsilon_x$
 $y = Y_0 + b + \epsilon_y$



Consequence

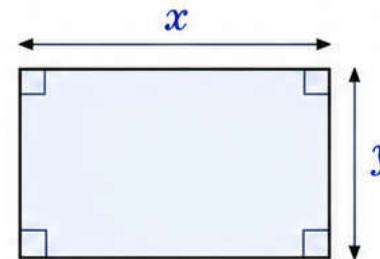
Deviations from the means contain the same systematic part:

$$x - X_0 = b + \epsilon_x \approx y - Y_0 = b + \epsilon_y$$

Common systematic contribution

Correlated deviations → **NOT** statistically independent

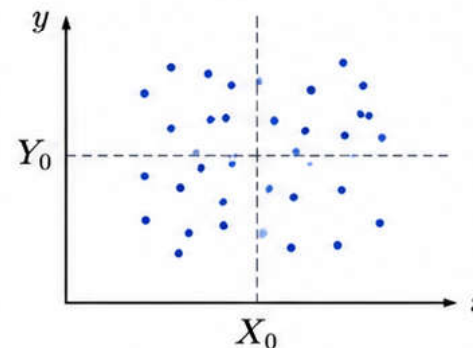
Visual picture



What we want:

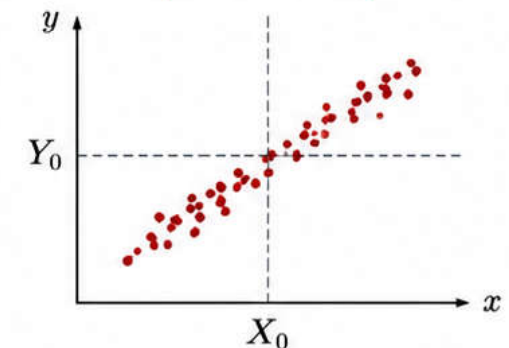
Independent random errors in x and y .

Independent case (ideal)



No correlation:
knowing x gives no information about y (and vice versa).

Non-independent case (common bias b)



Correlation present:
knowing x helps to predict y .



Key message: If measurements share a common systematic bias, their deviations are correlated $\Rightarrow$ they are **NOT** statistically independent.

11. Linear Functions of Measured Quantities

GENERAL IDEA

Many results in experiments can be written as a **linear function** of measured quantities.

$$Q = a + bX + cY + dZ + \dots$$

where $a, b, c, d, \dots$ are constants (known numbers),
and $X, Y, Z, \dots$ are measured quantities.

THIS INCLUDES

1 Addition

$$Q = X + Y$$

Example: total length

$$L = L_1 + L_2$$

2 Subtraction

$$Q = X - Y$$

Example: difference of readings

$$\Delta T = T_{\text{hot}} - T_{\text{cold}}$$

3 Multiplication by a constant

$$Q = bX$$

Example: unit conversion

$$s = 1000 t \text{ (ms from s)}$$



Key message: Linear relationships include **addition**, **subtraction**, and **multiplication by a constant**.

12. Mean of a Linear Combination

LINEAR MODEL

Consider a **linear function** of measured quantities:

$$Q = a + bX + cY + dZ + \dots$$

where $a, b, c, d, \dots$ are constants,
and $X, Y, Z, \dots$ are measured quantities.

MEAN VALUE

Let $X_0, Y_0, Z_0, \dots$ be the (true) mean values of $X, Y, Z, \dots$

Then the mean of Q is

$$Q_0 = a + bX_0 + cY_0 + dZ_0 + \dots$$

In other words,

take the mean of each measured quantity and substitute into the formula.



KEY MESSAGE

**Means
propagate
algebraically.**

No special treatment is needed
for the **mean**—just use algebra.



For linear functions, the mean of the result is obtained by the **same linear combination of the means**.

13. Uncertainty of a Linear Combination

ASSUMPTION

The input quantities $X, Y, Z, \dots$ are **statistically independent**.

$$Q = a + bX + cY + dZ + \dots$$

UNCERTAINTY (STANDARD DEVIATION)

For statistically **independent** inputs, the (standard) uncertainty of Q is

$$\delta Q = \sqrt{b^2 (\delta X)^2 + c^2 (\delta Y)^2 + d^2 (\delta Z)^2 + \dots}$$

or, equivalently,

$$(\delta Q)^2 = b^2 (\delta X)^2 + c^2 (\delta Y)^2 + d^2 (\delta Z)^2 + \dots$$



KEY MESSAGE

Independent
uncertainty contributions
combine in **quadrature**.

We add **variances** (squares of uncertainties), not uncertainties.



Interpretation

Each input X_i contributes an uncertainty δX_i to Q .

In the linear model, it is scaled by the coefficient b_i (**sensitivity**): contribution to δQ is $|b_i| \delta X_i$.

Because the inputs are independent, these contributions add as **perpendicular** (orthogonal) components in “uncertainty space” → **add in quadrature**.

Not to do

Do **NOT** use

$|b| \delta X + |c| \delta Y + \dots$
(that would be worst-case).

14. Why Quadrature?

THE IDEA

For a linear combination of two independent quantities:

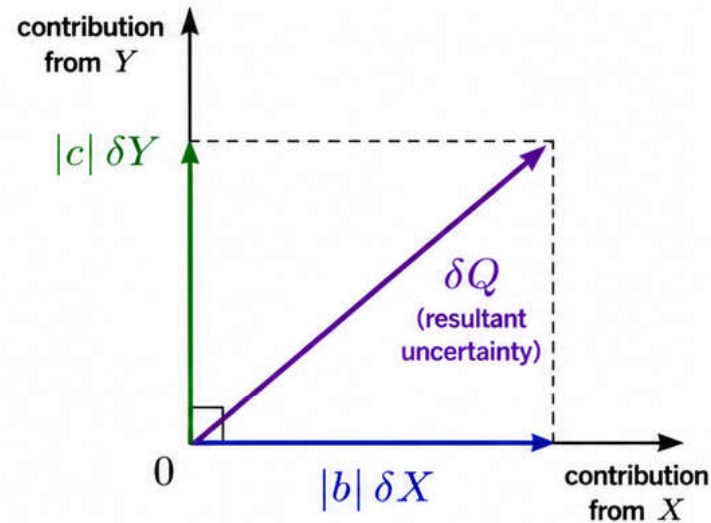
$$Q = bX + cY$$

the uncertainty is

$$(\delta Q)^2 = (b \delta X)^2 + (c \delta Y)^2$$

Independent uncertainty contributions combine in quadrature.

Pythagorean (orthogonal) picture



THIS IS NOT IT!

It is incorrect to add uncertainties linearly:

$$\delta Q = |b| \delta X + |c| \delta Y$$



This would overestimate the uncertainty (worst case).



INTUITION

Uncertainty contributions from independent inputs act in “**perpendicular directions**” in uncertainty space. Therefore, we use the Pythagorean theorem (**add in quadrature**), not simple arithmetic addition.

Generalization to more terms:

$$(\delta Q)^2 = (b \delta X)^2 + (c \delta Y)^2 + (d \delta Z)^2 + \dots$$



KEY TAKEAWAY

Quadrature comes from the **geometry** of independent contributions:

square each contribution, **add** the squares, then **take the square root**.

15. Addition

LINEAR MODEL

Addition is a special case of the linear model.

$$Q = X + Y$$

Here: $a = 0$, $b = 1$, $c = 1$.



KEY MESSAGE

When adding measurements:
uncertainties add in **quadrature**,
not linearly.



MEAN VALUE

Let X_0 and Y_0 be the true (mean) values of X and Y .
Then the mean of Q is

$$Q_0 = X_0 + Y_0$$

Means propagate algebraically.



UNCERTAINTY (INDEPENDENT INPUTS)

If X and Y are statistically independent, then the standard uncertainty of $Q = X + Y$ is

$$\delta Q = \sqrt{(\delta X)^2 + (\delta Y)^2}$$

or equivalently: $(\delta Q)^2 = (\delta X)^2 + (\delta Y)^2$

Independent uncertainty contributions combine in quadrature.



EXAMPLE (SYMBOLIC)

Suppose we have two independent measurements:

$$X = (X_0 \pm \delta X)$$

$$Y = (Y_0 \pm \delta Y)$$



For their sum $Q = X + Y$:

$$Q_0 = X_0 + Y_0$$

$$\delta Q = \sqrt{(\delta X)^2 + (\delta Y)^2}$$

Values add.
Uncertainties
add in **quadrature**.

16. Example: Absolute Pressure

PHYSICAL SITUATION

Absolute pressure is the sum of gauge (relative) pressure and atmospheric pressure.

LINEAR MODEL

$$P = P_{\text{rel}} + P_{\text{atm}}$$

MEAN VALUE

The mean (true) value is

$$P_0 = P_{\text{rel},0} + P_{\text{atm},0}$$



KEY MESSAGE

When adding independent quantities, uncertainties add in **quadrature**, not linearly.



UNCERTAINTY (INDEPENDENT INPUTS)

If P_{rel} and P_{atm} are statistically independent, then the standard uncertainty of $P = P_{\text{rel}} + P_{\text{atm}}$ is

$$\delta P = \sqrt{(\delta P_{\text{rel}})^2 + (\delta P_{\text{atm}})^2}$$

Independent uncertainty contributions combine in quadrature.



NOT LINEAR!

It is incorrect to add uncertainties linearly:

$$\delta P \neq \delta P_{\text{rel}} + \delta P_{\text{atm}}$$

That would overestimate the uncertainty (worst case).

MEASURED QUANTITIES

$$P_{\text{rel}} = (0.475 \pm 0.004) \text{ bar}$$

$$P_{\text{atm}} = (0.988 \pm 0.002) \text{ bar}$$

RELATION

Then the absolute pressure is

$$P = P_{\text{rel}} + P_{\text{atm}}$$



NEXT SLIDE

We calculate the mean value and the uncertainty.

18. Subtraction

LINEAR MODEL

Subtraction is also a special case of the linear model.

$$Q = X - Y$$

μ MEAN VALUE

If X_0 and Y_0 are the true (mean) values of X and Y , then the mean of $Q = X - Y$ is

$$Q_0 = X_0 - Y_0$$

Values subtract.



UNCERTAINTY (INDEPENDENT INPUTS)

If X and Y are statistically independent, then the standard uncertainty of $Q = X - Y$ is

$$\delta Q = \sqrt{(\delta X)^2 + (\delta Y)^2}$$

or equivalently:

$$(\delta Q)^2 = (\delta X)^2 + (\delta Y)^2$$

Uncertainty contributions combine in quadrature.



KEY MESSAGE

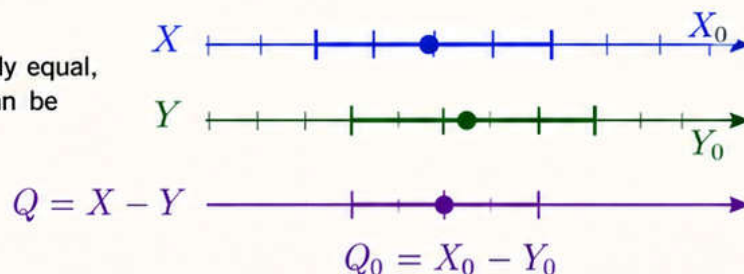
Values subtract —
uncertainties do **not**.

Why?

Uncertainties describe the spread (variance) of possible values. Their contributions are always non-negative, so they add in quadrature, never subtract.

WHAT THIS MEANS

Even if two measured values are nearly equal, the uncertainty of their difference can be significant.



The uncertainty of Q depends on the sizes of δX and δY , not on how close X_0 and Y_0 are.

NOT LINEAR!

It is incorrect to subtract uncertainties:

$$\delta Q \neq |\delta X - \delta Y|$$



That would underestimate the uncertainty (best case) and is not statistically valid.



SUMMARY

For subtraction of independent quantities:

$$Q_0 = X_0 - Y_0$$

(values subtract)

$$\delta Q = \sqrt{(\delta X)^2 + (\delta Y)^2}$$

(uncertainties add in quadrature)

19. Example: Differential Pressure

PHYSICAL SITUATION

The differential pressure between a container (cont) and the atmosphere is the difference of two absolute pressures.

LINEAR MODEL

$$P_{\text{diff}} = P_{\text{cont}} - P_{\text{atm}}$$

MEAN VALUE

The mean (true) value is

$$P_{\text{diff},0} = P_{\text{cont},0} - P_{\text{atm},0}$$

MEASURED QUANTITIES

$$P_{\text{cont}} = (P_{\text{cont},0} \pm \delta P_{\text{cont}}) \text{ bar}$$

$$P_{\text{atm}} = (P_{\text{atm},0} \pm \delta P_{\text{atm}}) \text{ bar}$$



UNCERTAINTY (INDEPENDENT INPUTS)

If P_{cont} and P_{atm} are statistically independent, then the standard uncertainty of $P_{\text{diff}} = P_{\text{cont}} - P_{\text{atm}}$ is

$$\delta P_{\text{diff}} = \sqrt{(\delta P_{\text{cont}})^2 + (\delta P_{\text{atm}})^2}$$

or equivalently:

$$(\delta P_{\text{diff}})^2 = (\delta P_{\text{cont}})^2 + (\delta P_{\text{atm}})^2$$

Independent uncertainty contributions combine in quadrature.



KEY MESSAGE

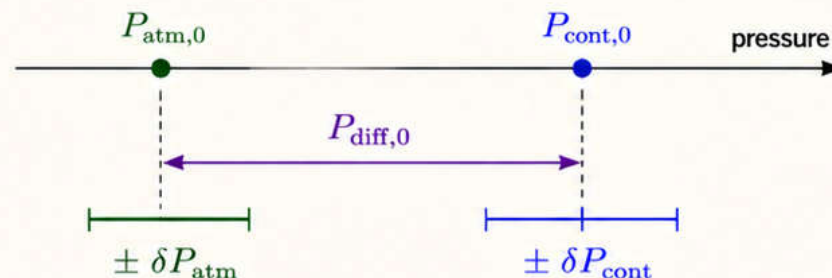
Values subtract — uncertainties do not.

Why?

Uncertainties describe the spread (variance) of possible values. Their contributions are always non-negative, so they add in quadrature, never subtract.

INTERPRETATION (GRAPHICAL)

Differential pressure is the horizontal distance between two absolute pressure values.



NOT LINEAR!

It is incorrect to subtract uncertainties:

$$\delta P_{\text{diff}} \neq |\delta P_{\text{cont}} - \delta P_{\text{atm}}|$$



That would underestimate the uncertainty (best case) and is not statistically valid.



SUMMARY

For the differential pressure of two independent absolute pressure measurements:

$$P_{\text{diff},0} = P_{\text{cont},0} - P_{\text{atm},0}$$

(values subtract)

$$\delta P_{\text{diff}} = \sqrt{(\delta P_{\text{cont}})^2 + (\delta P_{\text{atm}})^2}$$

(uncertainties add in quadrature)

21. What changes for a nonlinear function?

MAIN IDEA

Consider a nonlinear relationship

$$Q = f(X)$$

For the same input uncertainty δX , the resulting output uncertainty δQ depends on **where we are on the curve**.



KEY MESSAGE

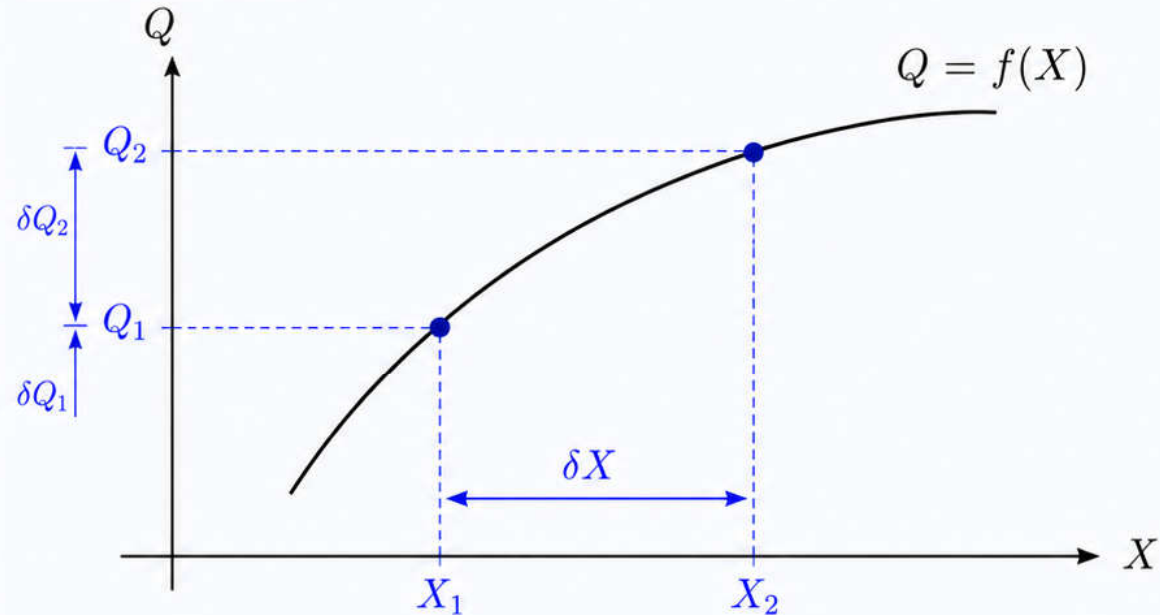
In a nonlinear function, the same change in input **does not produce the same change in output** everywhere.



WHY IT MATTERS

Uncertainty propagation depends on the **local behavior (slope)** of the function.

VISUAL PICTURE



CONCLUSION



The output uncertainty δQ is not determined by δX alone, but also by the **location (value of X)** through the **slope** of the function at that point.

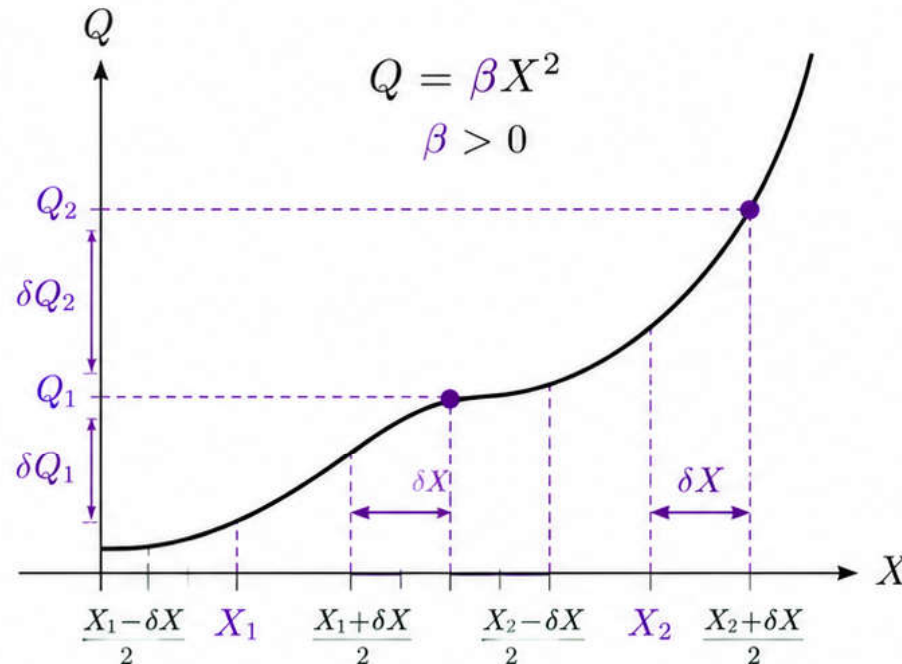
22. Example: quadratic dependence

MODEL

Consider a quadratic function:

$$Q = \beta X^2$$

This is a parabola opening upward
(here $\beta > 0$).



KEY MESSAGE

Take the same horizontal interval δX in two different places on the curve.



The resulting vertical intervals δQ_1 and δQ_2 are different.



The **sensitivity depends** on X .

INTUITION



Near $X = 0$ the parabola is flat $\rightarrow$ a given δX causes a small change in Q .

Farther from zero the curve is steeper $\rightarrow$ the same δX causes a larger change in Q .

TAKEAWAY



In nonlinear functions, equal input uncertainties do not lead to equal output uncertainties.

The output uncertainty δQ depends on **where we are on the curve** (i.e., on the value of X).

23. Local linearization

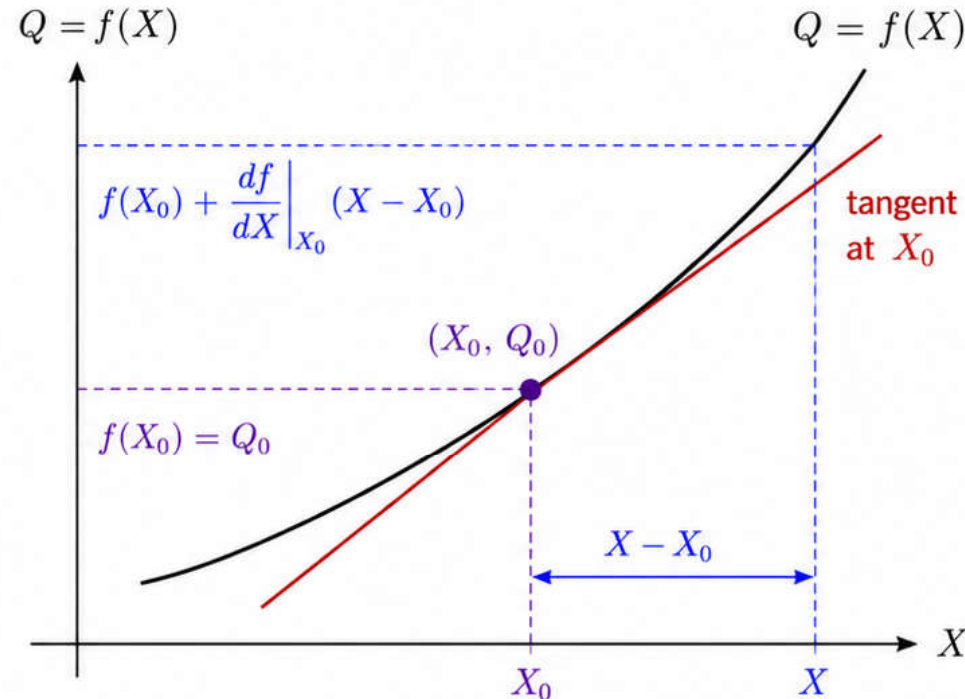
MODEL

For a nonlinear function $Q = f(X)$, near a point X_0 we can approximate it by its tangent line.

$$f(X) \approx f(X_0) + \left. \frac{df}{dX} \right|_{X_0} (X - X_0)$$

where

- $f(X_0) = Q_0$ is the function value at X_0 ,
- $\left. \frac{df}{dX} \right|_{X_0}$ is the slope (derivative) at X_0 .



KEY MESSAGE

Near X_0 , the nonlinear function $f(X)$ is well approximated by a linear function: its **tangent line**.



This **local linearization** is the foundation of uncertainty propagation for nonlinear functions.

INTUITION



Over a small interval around X_0 , the curve looks almost like a straight line. Therefore, small changes ΔX in X produce approximately linear changes ΔQ in Q .

TAKEAWAY



We can replace the nonlinear function by its **tangent line** near X_0 to relate small input changes to output changes.

$$f(X) \approx f(X_0) + f'(X_0)(X - X_0)$$

24. The derivative as sensitivity

MODEL

For a nonlinear function $Q = f(X)$, near a point X_0 we use the local linearization:

$$f(X) \approx f(X_0) + \left. \frac{df}{dX} \right|_{X_0} (X - X_0)$$

Let $Q_0 = f(X_0)$. Then

$$\Delta Q \approx \left. \frac{df}{dX} \right|_{X_0} \Delta X$$

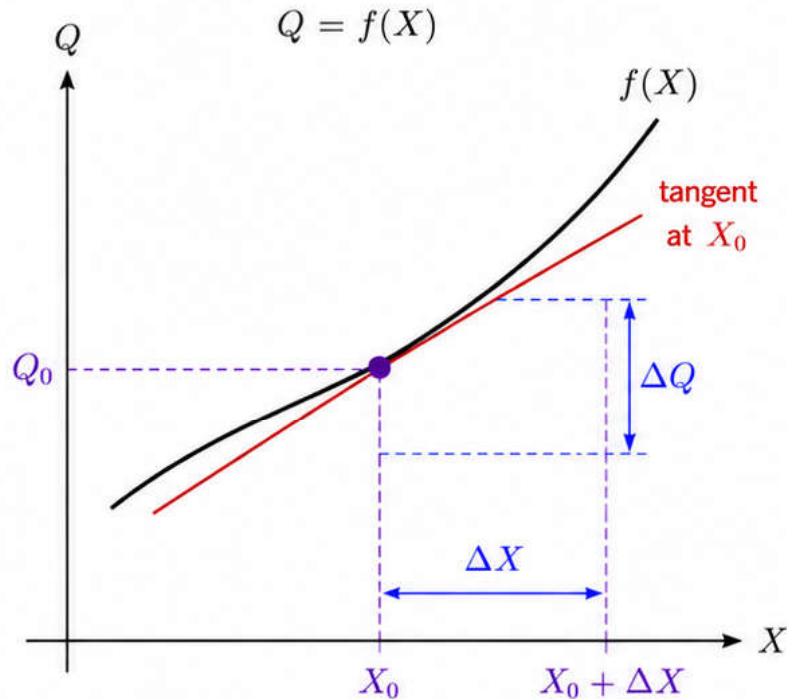


DEFINITION

Define the sensitivity coefficient

$$c_X = \left. \frac{\partial Q}{\partial X} \right|_0 = \left. \frac{df}{dX} \right|_{X_0}$$

(the derivative of Q with respect to X evaluated at X_0).



KEY MESSAGE

The derivative (sensitivity coefficient) tells us **how strongly** the output Q reacts to a change in the input X .

$$\Delta Q \approx c_X \Delta X$$

For uncertainties (using absolute values):

$$\delta Q \approx |c_X| \delta X$$



PHYSICAL INTERPRETATION

The derivative is the local gain of the system:

- If $|c_X|$ is large $\Rightarrow$ small input changes produce **large output changes**.
- If $|c_X|$ is small $\Rightarrow$ the output is almost **insensitive** to input changes.
- The sign of c_X indicates the direction of change (increase or decrease).



TAKEAWAY

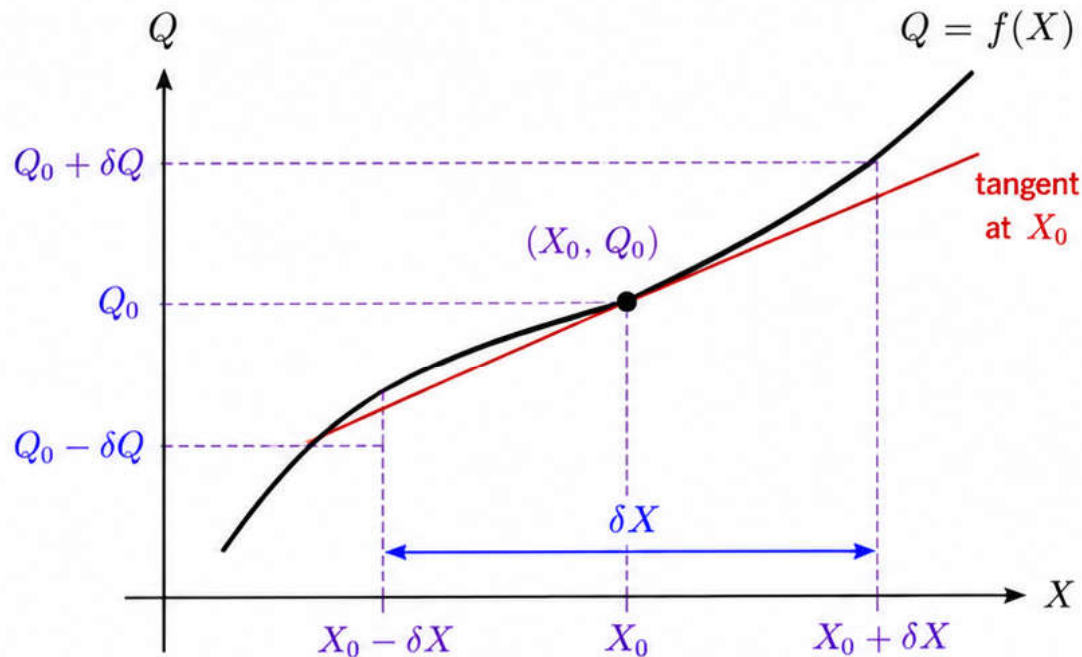
The derivative evaluated at the operating point X_0 is the **sensitivity**:

$$\delta Q \approx |c_X| \delta X \quad \text{where} \quad c_X = \left. \frac{\partial Q}{\partial X} \right|_0$$

It quantifies the **local response** of Q to a small change in X .

25. Graphical meaning of uncertainty propagation

VISUALIZATION



INPUT UNCERTAINTY

$$X_0 \pm \delta X$$



TANGENT (LINEAR APPROXIMATION)

$$\text{slope} = \left. \frac{dQ}{dX} \right|_{X_0} = c_X$$



OUTPUT UNCERTAINTY

$$Q_0 \pm \delta Q$$

WHAT THE PLOT SHOWS

- A symmetric input uncertainty $X_0 \pm \delta X$ (horizontal interval) is given.
- Through the tangent (local linear model), this maps to an output interval $Q_0 \pm \delta Q$ (vertical interval).
- The size of δQ depends on the slope of the function at X_0 .
- The nonlinear curve is replaced locally by its tangent to propagate uncertainty.



KEY IDEA

Uncertainty in the **input** (horizontal) is transformed into uncertainty in the **output** (vertical) by the local **slope** of the function at the operating point.

LINEAR RELATION (LOCALLY)

$$\delta Q \approx \left| \left. \frac{dQ}{dX} \right|_{X_0} \right| \delta X \quad (= |c_X| \delta X)$$

26. Example: area of a circle

MODEL

The area of a circle with radius R is

$$A = \pi R^2$$

We want to determine how an uncertainty in R affects the area A .



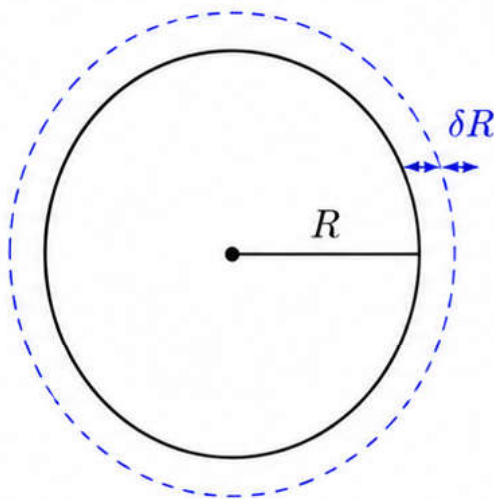
SENSITIVITY

The sensitivity of A with respect to R is

$$\frac{dA}{dR} = \frac{d}{dR} (\pi R^2) = 2\pi R$$

This tells us how strongly the area changes when the radius changes.

GEOMETRIC VIEW



— radius R
- - - radius $R + \delta R$

Radius: $R \pm \delta R$



Area: $A \pm \delta A$

LINEAR APPROXIMATION

For a small change δR around R_0 , the change in area is approximated by the tangent (linear model):

$$\delta A \approx \left| \frac{dA}{dR} \right|_{R_0} \delta R$$

Substitute the derivative:

$$\delta A \approx (2\pi R_0) \delta R$$



RESULT

$$\delta A \approx 2\pi R \delta R$$

(absolute uncertainties)



CONCLUSION

A small uncertainty in the radius δR causes an **area uncertainty δA proportional to the circumference $2\pi R$.**

The larger the radius R , the larger the effect of the same δR on the area.

$$\delta A \propto R$$

27. Same absolute uncertainty, different effect

SITUATION

We consider the area of a circle

$$A = \pi R^2.$$

From the previous slide:

$$\delta A \approx 2\pi R \delta R$$

Take the same absolute uncertainty in radius:

$$\delta R = 0.01 \text{ mm}$$

RELATIVE UNCERTAINTY

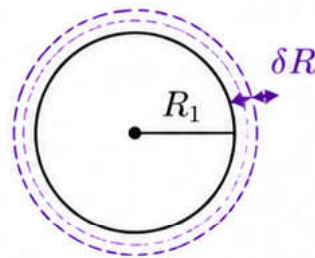
The relative uncertainty of the area is

$$\frac{\delta A}{A} = \frac{2\pi R \delta R}{\pi R^2} = \frac{2 \delta R}{R}$$

→ It is inversely proportional to R .
Larger radius $\Rightarrow$ smaller relative uncertainty.

EXAMPLE 1: SMALL RADIUS

$$R_1 = (0.5 \pm 0.01) \text{ mm}$$



$$R_1 = 0.5 \text{ mm} \\ \delta R = 0.01 \text{ mm}$$



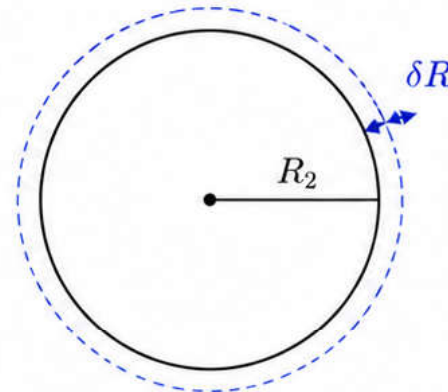
$$A_1 = \pi R_1^2 = \pi(0.5)^2 = 0.785 \text{ mm}^2$$

$$\delta A_1 \approx 2\pi R_1 \delta R = 2\pi(0.5)(0.01) = 0.0314 \text{ mm}^2$$

$$\frac{\delta A_1}{A_1} \approx \frac{0.0314}{0.785} = 0.040 \text{ (4.0\%)}$$

EXAMPLE 2: LARGE RADIUS

$$R_2 = (5.0 \pm 0.01) \text{ mm}$$



$$R_2 = 5.0 \text{ mm} \\ \delta R = 0.01 \text{ mm}$$



$$A_2 = \pi R_2^2 = \pi(5.0)^2 = 78.54 \text{ mm}^2$$

$$\delta A_2 \approx 2\pi R_2 \delta R = 2\pi(5.0)(0.01) = 0.314 \text{ mm}^2$$

$$\frac{\delta A_2}{A_2} \approx \frac{0.314}{78.54} = 0.0040 \text{ (0.40\%)}$$

COMPARISON

	Small radius (0.5 mm)	Large radius (5.0 mm)
δR (mm)	0.01	0.01
Area A (mm ²)	0.785	78.54
Area uncertainty δA (mm ²)	0.0314	0.314
Relative uncertainty $\delta A/A$	4.0%	0.40%



The same absolute uncertainty δR produces a **much larger** relative effect for **smaller** R .

TAKEAWAY



For $A = \pi R^2$:

- $\delta A \propto R$
- $\frac{\delta A}{A} \propto \frac{1}{R}$

Smaller radii are more sensitive!

29. General law of uncertainty propagation



MAIN RESULT (FOR INDEPENDENT INPUTS)

If the input quantities are independent, the (combined standard) uncertainty of the output is given by the **root-sum-of-squares** of the individual contributions:

$$\delta Q = \sqrt{\left(\frac{\partial Q}{\partial X} \delta X\right)^2 + \left(\frac{\partial Q}{\partial Y} \delta Y\right)^2 + \left(\frac{\partial Q}{\partial Z} \delta Z\right)^2 + \dots}$$

MODEL

We have a quantity of interest

$$Q = f(X, Y, Z, \dots)$$

depending on several **independent** input quantities $X, Y, Z, \dots$,

ASSUMPTIONS

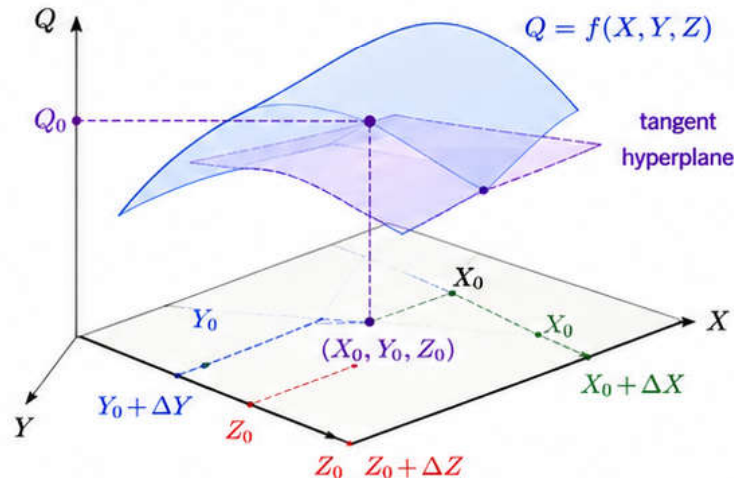
- The input quantities are **independent** (uncorrelated).
- Uncertainties are small, so the local linear approximation is valid.
- Only first-order terms are retained.

INDEPENDENCE $\Rightarrow$ no covariance terms (cross terms vanish)

DERIVATION IDEA: TANGENT HYPERPLANE

Near the operating point $(X_0, Y_0, Z_0, \dots)$:

$$Q \approx Q_0 + \frac{\partial Q}{\partial X} \Delta X + \frac{\partial Q}{\partial Y} \Delta Y + \frac{\partial Q}{\partial Z} \Delta Z + \dots$$



UNCERTAINTY PROPAGATION (INDEPENDENT INPUTS)

Taking absolute values and applying the law of propagation for **independent inputs**:

$$\delta Q = \sqrt{\underbrace{\left(\frac{\partial Q}{\partial X} \delta X\right)^2}_{\text{contribution from } X} + \underbrace{\left(\frac{\partial Q}{\partial Y} \delta Y\right)^2}_{\text{contribution from } Y} + \underbrace{\left(\frac{\partial Q}{\partial Z} \delta Z\right)^2}_{\text{contribution from } Z} + \dots}$$

WHAT IT MEANS

- Each input contributes to the uncertainty of Q through its **sensitivity** (partial derivative) and its own uncertainty.
- Contributions add in **quadrature** (root-sum-of-squares).
- Larger sensitivity or larger input uncertainty $\Rightarrow$ larger impact on the output uncertainty.



KEY TAKEAWAY

For a nonlinear model with independent inputs, the combined uncertainty is the square root of the sum of the squared individual contributions:

$$\delta Q = \sqrt{\sum_i \left(\frac{\partial Q}{\partial X_i} \delta X_i\right)^2} \quad (\text{independent inputs})$$

30. How to use the formula

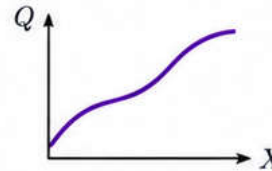
A practical **four-step workflow** for uncertainty propagation in nonlinear models.

1

WRITE THE MEASUREMENT MODEL

Express the output quantity as a function of the inputs.

$$Q = f(X, Y, Z, \dots)$$



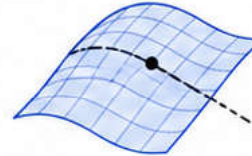
- Identify the output quantity of interest.
- List all input quantities that affect it.
- Write the functional relationship.

2

CALCULATE THE PARTIAL DERIVATIVES

Determine the sensitivity of Q to each input.

$$\frac{\partial Q}{\partial X}, \frac{\partial Q}{\partial Y}, \frac{\partial Q}{\partial Z}, \dots$$



- Compute partial derivatives at the operating (nominal) values.
- These are the sensitivity coefficients $c_X = \left| \frac{\partial Q}{\partial X} \right|$, $c_Y = \left| \frac{\partial Q}{\partial Y} \right|$, $\dots$

3

MULTIPLY EACH SENSITIVITY BY ITS UNCERTAINTY

Compute the individual contributions.

$$u_X = \left| \frac{\partial Q}{\partial X} \right| \delta X, \quad u_Y = \left| \frac{\partial Q}{\partial Y} \right| \delta Y, \quad u_Z = \left| \frac{\partial Q}{\partial Z} \right| \delta Z, \quad \dots$$



- Multiply the absolute value of each partial derivative by the standard uncertainty of its input.
- Each result is an uncertainty contribution.

4

COMBINE INDEPENDENT CONTRIBUTIONS IN QUADRATURE

Obtain the combined standard uncertainty of the output.

$$\delta Q = \sqrt{u_X^2 + u_Y^2 + u_Z^2 + \dots}$$

$$\boxed{u_X^2} + \boxed{u_Y^2} + \boxed{u_Z^2} + \dots = \delta Q^2$$

- Add the squared contributions.
- Take the square root to get δQ .
- This is valid for independent inputs.

31. Example: uncertainty propagation in a nonlinear model



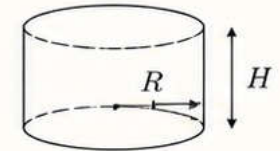
PROBLEM

A cylindrical tank has measured radius R and height H .
The volume is $V = \pi R^2 H$. Given uncertainties in R and H , determine the uncertainty in V .

Given (independent inputs):

$$R = 0.500 \text{ m} \pm 0.005 \text{ m}$$

$$H = 1.200 \text{ m} \pm 0.010 \text{ m}$$



1 WRITE THE MODEL

Volume of a cylinder:

$$V = \pi R^2 H$$

Nominal values:

$$R_0 = 0.500 \text{ m}$$

$$H_0 = 1.200 \text{ m}$$



Goal:

Find δV (combined standard uncertainty in V).

2 CALCULATE PARTIAL DERIVATIVES (SENSITIVITIES)

Compute at the nominal values:

$$\frac{\partial V}{\partial R} = 2\pi R H$$

$$\text{at nominal: } 2\pi(0.500)(1.200) = 3.770 \text{ m}^2$$

$$\frac{\partial V}{\partial H} = \pi R^2$$

$$\text{at nominal: } \pi(0.500)^2 = 0.7854 \text{ m}^2$$

Sensitivities:

$$c_R = \frac{\partial V}{\partial R} = 3.770 \text{ m}^2$$

$$c_H = \frac{\partial V}{\partial H} = 0.7854 \text{ m}^2$$

3 MULTIPLY BY INPUT UNCERTAINTIES

Standard uncertainties:

$$u_R = 0.005 \text{ m}$$

$$u_H = 0.010 \text{ m}$$

Individual contributions:

Input	Formula	Value	Unit
R	$u_V(R) = \left \frac{\partial V}{\partial R} \right u_R$	$(3.770)(0.005) = 0.01885$	m^3
H	$u_V(H) = \left \frac{\partial V}{\partial H} \right u_H$	$(0.7854)(0.010) = 0.007854$	m^3



These are the standard uncertainty contributions from R and H .

4 COMBINE IN QUADRATURE

Since inputs are independent:

$$\begin{aligned} \delta V &= \sqrt{u_V(R)^2 + u_V(H)^2} \\ &= \sqrt{(0.01885)^2 + (0.007854)^2} \\ &= 0.02040 \text{ m}^3 \end{aligned}$$



RESULT

$$V = \pi R^2 H = 0.94248 \text{ m}^3$$

$$\delta V = 0.0204 \text{ m}^3$$

(standard uncertainty)



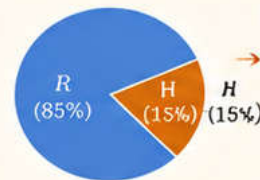
INTERPRETATION

At the 1σ (68%) confidence level, the true volume is expected to lie in

$$0.94248 \pm 0.0204 \text{ m}^3$$

i.e., between 0.9221 and 0.9629 m^3 .

RELATIVE CONTRIBUTIONS (optional)



Fractional contributions to variance:

$$\text{From } R: \frac{u_V(R)^2}{\delta V^2} = 85.3\%$$

$$\text{From } H: \frac{u_V(H)^2}{\delta V^2} = 14.7\%$$



KEY TAKEAWAY

Use the general law of uncertainty propagation:

$$\delta V = \sqrt{\left(\frac{\partial V}{\partial R} u_R \right)^2 + \left(\frac{\partial V}{\partial H} u_H \right)^2}$$

to determine how input uncertainties affect the output.

32. Product

PRODUCT OF TWO INDEPENDENT VARIABLES

Consider a quantity that is the product of two independent variables:

$$Q = X Y$$

GENERAL PROPAGATION LAW

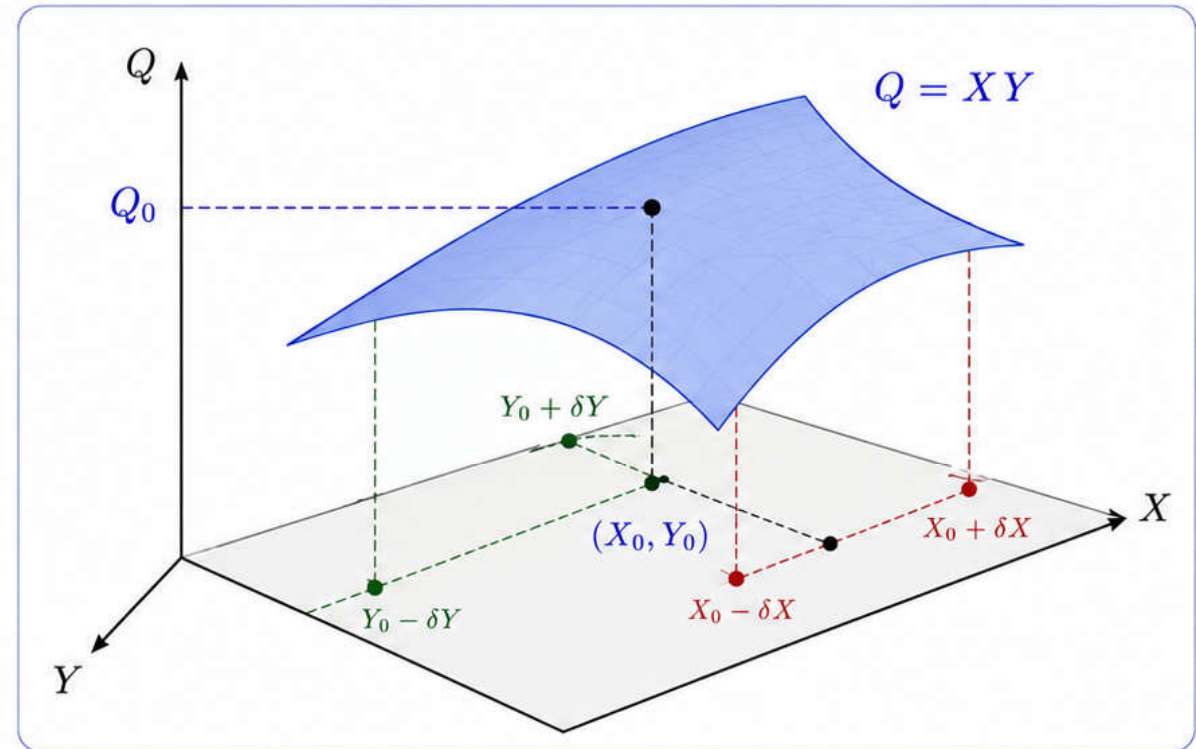
From the general uncertainty propagation formula (for independent inputs):

$$\delta Q = \sqrt{\left(\frac{\partial Q}{\partial X} \delta X\right)^2 + \left(\frac{\partial Q}{\partial Y} \delta Y\right)^2}$$

For $Q = XY$:

$$\frac{\partial Q}{\partial X} = Y \quad \text{and} \quad \frac{\partial Q}{\partial Y} = X$$

$$\rightarrow \delta Q = \sqrt{Y^2(\delta X)^2 + X^2(\delta Y)^2}$$



KEY POINT

The absolute uncertainty of a product depends on both the value of each variable and its own uncertainty.

33. Relative uncertainty of a product

PRODUCT

For the product of two independent variables:

$$Q = X Y$$

and from the previous slide:

$$\delta Q = \sqrt{Y^2 (\delta X)^2 + X^2 (\delta Y)^2}$$

RELATIVE UNCERTAINTY

Divide both sides by $Q = XY$:

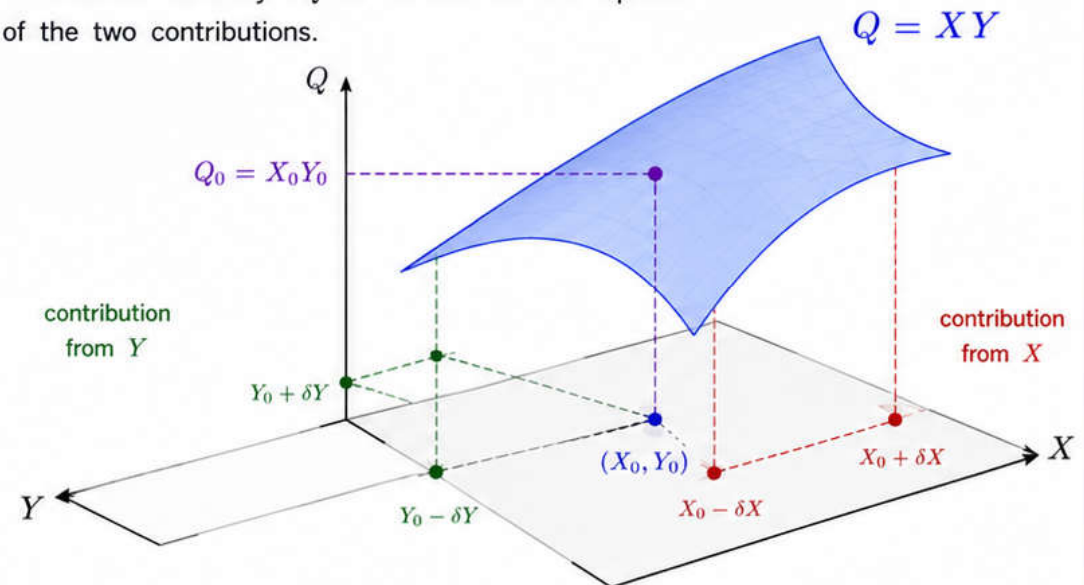
$$\begin{aligned}\frac{\delta Q}{Q} &= \frac{\sqrt{Y^2 (\delta X)^2 + X^2 (\delta Y)^2}}{XY} \\ &= \sqrt{\frac{Y^2 (\delta X)^2}{X^2 Y^2} + \frac{X^2 (\delta Y)^2}{X^2 Y^2}}\end{aligned}$$

$$\left(\frac{\delta Q}{Q}\right)^2 = \left(\frac{\delta X}{X}\right)^2 + \left(\frac{\delta Y}{Y}\right)^2$$

→ GENERAL RESULT
FOR PRODUCTS

GEOMETRIC INTERPRETATION

At the nominal point (X_0, Y_0) the surface $Q = XY$ has slopes Y_0 in the X direction and X_0 in the Y direction. The total uncertainty δQ is the root-sum-of-squares of the two contributions.



KEY TAKEAWAY

For products, relative uncertainties combine in quadrature:

For products, relative uncertainties combine in quadrature.

IMPLICATIONS

- Both variables contribute positively.
- Larger relative uncertainty in either variable increases the relative uncertainty of the product.
- If one term dominates, the result is approximately equal to the larger relative uncertainty.

34. Quotient

QUOTIENT OF TWO INDEPENDENT VARIABLES

Consider a quantity that is the quotient of two independent variables:

$$Q = \frac{Y}{X}$$

(where $X \neq 0$)

UNCERTAINTY PROPAGATION

From the general propagation law (independent inputs):

$$\delta Q = \sqrt{\left(\frac{\partial Q}{\partial X} \delta X\right)^2 + \left(\frac{\partial Q}{\partial Y} \delta Y\right)^2}$$

For $Q = \frac{Y}{X}$:

$$\frac{\partial Q}{\partial X} = -\frac{Y}{X^2} = -\frac{Q}{X} \quad \frac{\partial Q}{\partial Y} = \frac{1}{X} = \frac{Q}{Y}$$

$$\Rightarrow \delta Q = |Q| \sqrt{\left(\frac{\delta X}{X}\right)^2 + \left(\frac{\delta Y}{Y}\right)^2}$$



IMPORTANT CONTRAST

Although division is involved,

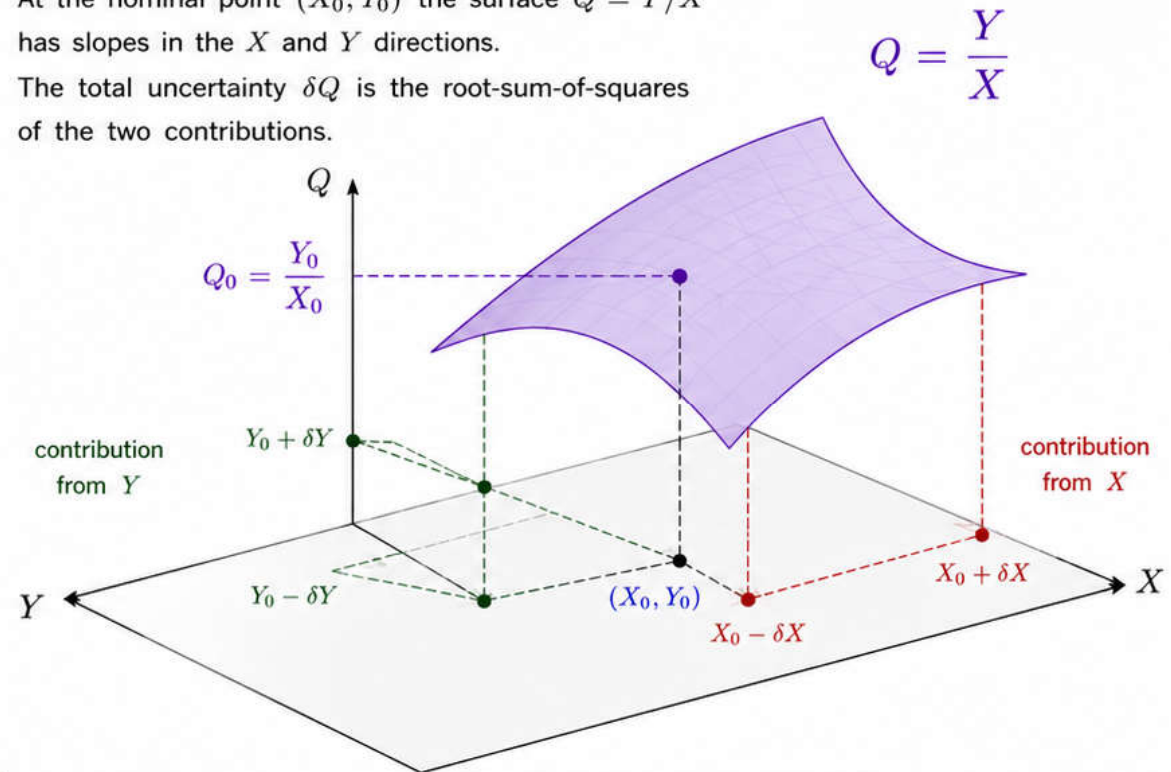
Division does not mean subtracting uncertainties.

Relative uncertainties still combine in quadrature.

GEOMETRIC INTERPRETATION

At the nominal point (X_0, Y_0) the surface $Q = Y/X$ has slopes in the X and Y directions.

The total uncertainty δQ is the root-sum-of-squares of the two contributions.



KEY TAKEAWAY

For a quotient, the relative uncertainty of the result is given by

$$\left(\frac{\delta Q}{Q}\right)^2 = \left(\frac{\delta X}{X}\right)^2 + \left(\frac{\delta Y}{Y}\right)^2$$

35. Example: area of a rectangle

PROBLEM

Find the area of a rectangle given the measured side lengths with uncertainties.

MODEL

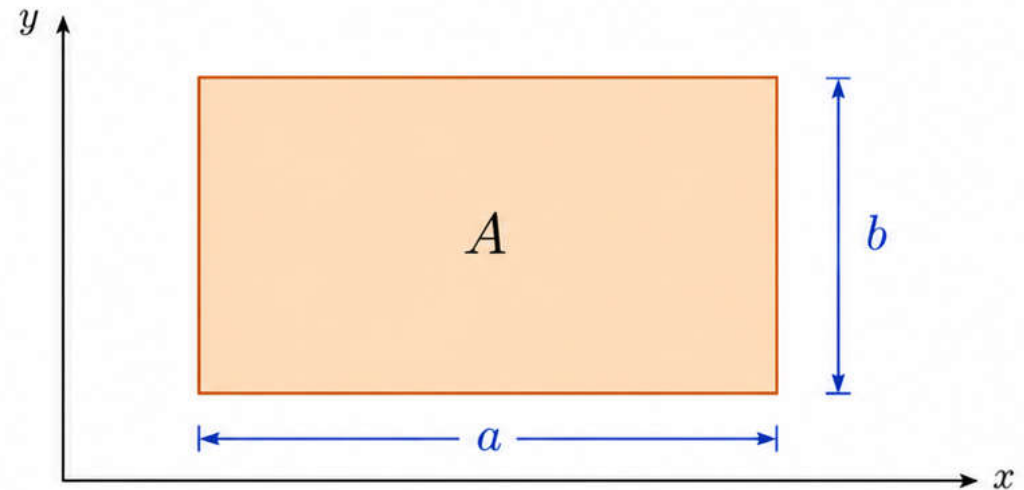
The area of a rectangle is the product of its sides:

$$A = a b$$

MEASURED QUANTITIES

- Side a (length): $a = (13.00 \pm 0.05) \text{ cm}$
- Side b (width): $b = (25.00 \pm 0.05) \text{ cm}$

GEOMETRIC REPRESENTATION



NOMINAL AREA

Using the nominal (central) values:

$$A = a b = (13.00 \text{ cm}) (25.00 \text{ cm}) = 325 \text{ cm}^2$$



KEY TAKEAWAY

For the area of a rectangle, the measured sides combine multiplicatively. The nominal area is simply the product of the nominal side lengths.

$$A = 325 \text{ cm}^2$$

36. Rectangle: uncertainty

UNCERTAINTY FORMULA

For the area $A = ab$ of a rectangle (product of independent variables):

$$\left(\frac{\delta A}{A}\right)^2 = \left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta b}{b}\right)^2$$

or, equivalently,

$$\delta A = A \sqrt{\left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta b}{b}\right)^2}$$

NUMERICAL EVALUATION

- $a = (13.00 \pm 0.05) \text{ cm}$, $\delta a = 0.05 \text{ cm}$
- $b = (25.00 \pm 0.05) \text{ cm}$, $\delta b = 0.05 \text{ cm}$
- $A = ab = 325 \text{ cm}^2$

Relative uncertainty:

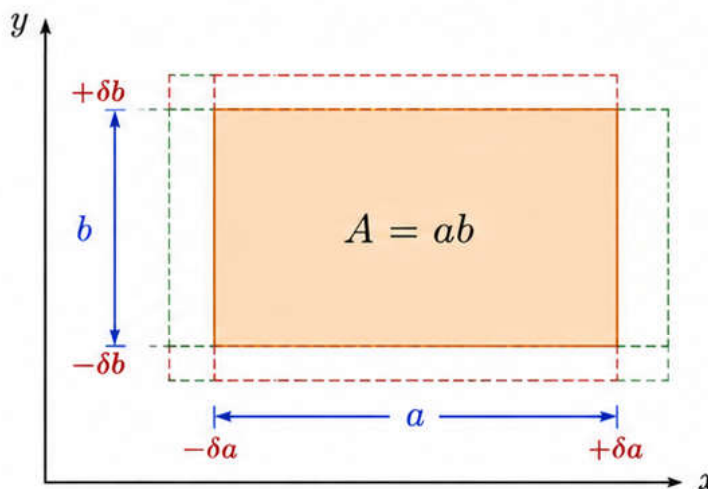
$$\left(\frac{\delta A}{A}\right)^2 = \left(\frac{0.05}{13.00}\right)^2 + \left(\frac{0.05}{25.00}\right)^2 = 2.393 \times 10^{-5}$$

$$\frac{\delta A}{A} = \sqrt{2.393 \times 10^{-5}} = 0.004891 = 0.489\%$$

$$\delta A = A \left(\frac{\delta A}{A}\right) = 325 \times 0.004891 = \mathbf{1.59 \text{ cm}^2}$$

VISUALIZATION OF CONTRIBUTIONS

Uncertainty in a (horizontal) and b (vertical) each produce a change in area.
The total uncertainty is the root-sum-of-squares of these two contributions.



Contribution from δa (width):

$$\Delta A_a \approx A \frac{\delta a}{a}$$

Changes area by a vertical "strip" of width δa .

Contribution from δb (height):

$$\Delta A_b \approx A \frac{\delta b}{b}$$

Changes area by a horizontal "strip" of height δb .

Total (combined) uncertainty:

$$\delta A = \sqrt{(\Delta A_a)^2 + (\Delta A_b)^2}$$

RESULT

Area of the rectangle with uncertainty: $A = (325 \pm 1.6) \text{ cm}^2$ (rounded)

Relative uncertainty: 0.489%



KEY TAKEAWAY

Each measured side contributes to the uncertainty of the area. Their effects add in quadrature (not linearly).

Here, the width uncertainty (δa) dominates because $\frac{\delta a}{a} > \frac{\delta b}{b}$.

Relative contributions to variance:

from a : **81.3%**

from b : **18.7%**

37. Power law

POWER LAW

Consider a quantity Q that depends on a single measured variable X through a power:

$$Q = X^n$$

where n is a (real) constant.

RESULT (POWER LAW)

For small uncertainties:

$$\delta Q \approx |n| X^{n-1} \delta X$$

Equivalently,

$$\delta Q \approx |n| Q \frac{\delta X}{X}$$

Relative form:

$$\frac{\delta Q}{Q} \approx |n| \frac{\delta X}{X}$$



IMPLICATIONS

- The relative uncertainty is multiplied by $|n|$.
- Higher powers amplify relative uncertainties.
- Very useful when a quantity depends on a measured variable through a power (e.g., area, volume, moments of inertia, ...).

DERIVATION FROM THE GENERAL PROPAGATION LAW

From the general law for one variable:

$$\delta Q = \left| \frac{dQ}{dX} \right| \delta X$$

For $Q = X^n$:

$$\frac{dQ}{dX} = nX^{n-1}$$

$$\Rightarrow \delta Q = |n| X^{n-1} \delta X$$

Divide by $Q = X^n$:

$$\frac{\delta Q}{Q} = |n| \frac{\delta X}{X}$$

$$\delta Q \approx |n| X^{n-1} \delta X = |n| Q \frac{\delta X}{X}$$

EXAMPLES

1. Area of a circle

$$A = \pi R^2 = (\pi) R^2$$

$$n = 2$$

$$\Rightarrow \frac{\delta A}{A} \approx 2 \frac{\delta R}{R}$$

2. Volume of a sphere

$$V = \frac{4}{3} \pi R^3 = \left(\frac{4}{3} \pi \right) R^3$$

$$n = 3$$

$$\Rightarrow \frac{\delta V}{V} \approx 3 \frac{\delta R}{R}$$



KEY TAKEAWAY

If $Q = X^n$, then for small uncertainties

$$\delta Q \approx |n| X^{n-1} \delta X \quad \text{and} \quad \frac{\delta Q}{Q} \approx |n| \frac{\delta X}{X}$$

This general rule directly prepares us to analyze quantities like the volume of a ping-pong ball, where $V \propto R^3$.

38. Measurement exercise: ping-pong ball

GOAL

Measure the diameter of a ping-pong ball repeatedly and analyze the data statistically.

MEASUREMENTS

Measure the diameter d of the ping-pong ball N times:

$$d_1, d_2, \dots, d_N$$

(Use the same orientation of the ball and caliper for all measurements.)



DETERMINE FROM THE DATA

- 1 **Histogram:**
Plot the distribution of the measured diameters.

- 2 **Sample mean (average):**

$$\bar{d} = \frac{1}{N} \sum_{i=1}^N d_i$$

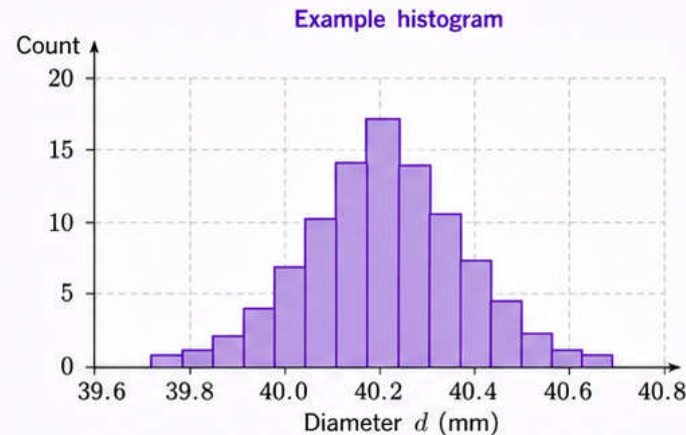
- 3 **Sample standard deviation (scatter of the data):**

$$s = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (d_i - \bar{d})^2}$$

- 4 **Standard deviation of the mean:**

$$s_{\bar{d}} = \frac{s}{\sqrt{N}}$$

(standard uncertainty of the mean).



NOTES

- Take at least $N \geq 20$ measurements.
- Record all raw data.
- The histogram reveals the shape of the distribution and possible outliers.
- s describes the scatter of individual measurements; $s_{\bar{d}}$ quantifies how well the mean diameter is known.



RESULT OF THIS EXERCISE

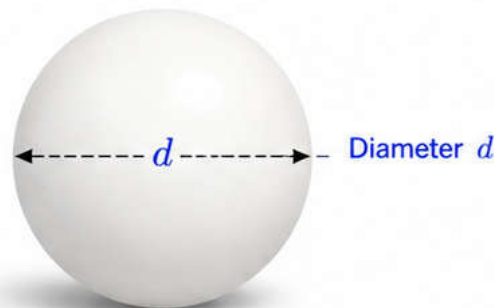
At the end you will have $\bar{d} \pm s_{\bar{d}}$ as the best estimate of the ball's diameter and its *standard uncertainty*.

39. From diameter to volume

MEASUREMENT MODEL

We measure the diameter d of a ping-pong ball.
The volume of a sphere in terms of diameter is

$$V = \frac{\pi}{6} d^3$$



SENSITIVITY (how sensitive is V to changes in d ?)

Differentiate $V = \frac{\pi}{6} d^3$ with respect to d :

$$\frac{dV}{dd} = \frac{\pi}{6} 3d^2 = \frac{\pi}{2} d^2$$

$$\rightarrow \frac{dV}{dd} = \frac{\pi}{2} d^2$$

UNCERTAINTY PROPAGATION

For small uncertainties, the uncertainty in V is approximated by

$$\delta V \approx \left| \frac{dV}{dd} \right| \delta d$$

Using $\frac{dV}{dd} = \frac{\pi}{2} d^2$:

$$\delta V = \frac{\pi}{2} d^2 \delta d$$

ELEGANT RELATIVE FORM

Divide $\delta V = \frac{\pi}{2} d^2 \delta d$ by $V = \frac{\pi}{6} d^3$:

$$\frac{\delta V}{V} = \frac{\frac{\pi}{2} d^2 \delta d}{\frac{\pi}{6} d^3} = \frac{\frac{\pi}{2} d^2 \delta d}{\frac{\pi}{6} d^3} = 3 \frac{\delta d}{d}$$

Thus,

$$\frac{\delta V}{V} = 3 \frac{\delta d}{d}$$

INTERPRETATION

- The relative uncertainty of the volume is three times the relative uncertainty of the diameter.
- Small relative errors in d lead to amplified (by a factor of 3) relative errors in V .

In typical measurements of a ping-pong ball, this corresponds to about **0.3%** relative uncertainty in the volume.

EXAMPLE (NUMERIC)

Suppose a measured diameter has standard uncertainty δd .

Then the volume uncertainty is

$$\delta V = \frac{\pi}{2} d^2 \delta d$$

or equivalently $\frac{\delta V}{V} = 3 \frac{\delta d}{d}$.

Measured d	Relative $\delta d/d$	Relative $\delta V/V$
40.20 mm	0.10%	0.30%

→ A 0.10% uncertainty in diameter leads to a 0.30% uncertainty in volume.



KEY TAKEAWAY

From diameter d to volume $V = \frac{\pi}{6} d^3$:

$$\frac{\delta V}{V} = 3 \frac{\delta d}{d}$$

This explains the $\sim 0.3\%$ relative uncertainty in volume observed in typical ping-pong ball measurements.

40. From mass and volume to density

DENSITY DEFINITION

The (mass) density ρ of the ping-pong ball is

$$\rho = \frac{m}{V}$$

where

m = mass of the ball,

V = volume of the ball.



NUMERICAL EXAMPLE

Typical measurement results (example):

Mass: $m = 2.70 \pm 0.01$ g

$$\frac{\delta m}{m} = \frac{0.01}{2.70} = 0.37\%$$

Volume: $V = 1.84 \pm 0.006$ cm³

$$\frac{\delta V}{V} = \frac{0.006}{1.84} = 0.33\%$$

Density:

$$\rho = \frac{m}{V} = \frac{2.70 \text{ g}}{1.84 \text{ cm}^3} = 1.47 \text{ g/cm}^3$$

Relative uncertainty of ρ :

$$\frac{\delta \rho}{\rho} = \sqrt{(0.37\%)^2 + (0.33\%)^2} = 0.50\%$$

Absolute uncertainty: $\delta \rho = 0.50\% \times 1.47 = 0.007$ g/cm³

Result: $\rho = (1.47 \pm 0.01) \text{ g/cm}^3$ (rounded)

UNCERTAINTY PROPAGATION

For a function of two independent variables m and V :

$$(\delta \rho)^2 = \left(\frac{\partial \rho}{\partial m} \delta m \right)^2 + \left(\frac{\partial \rho}{\partial V} \delta V \right)^2$$

For $\rho = \frac{m}{V}$:

$$\frac{\partial \rho}{\partial m} = \frac{1}{V}, \quad \frac{\partial \rho}{\partial V} = -\frac{m}{V^2} = -\frac{\rho}{V}$$



$$(\delta \rho)^2 = \left(\frac{\delta m}{V} \right)^2 + \left(\frac{\rho \delta V}{V} \right)^2$$

Divide by $\rho = \frac{m}{V}$:

$$\left(\frac{\delta \rho}{\rho} \right)^2 = \left(\frac{\delta m}{m} \right)^2 + \left(\frac{\delta V}{V} \right)^2$$

Relative-uncertainty form

For a quotient, relative uncertainties combine in quadrature.

GENERAL QUOTIENT RULE (RECAP)

For $Q = \frac{X}{Y}$, the relative uncertainty is

$$\left(\frac{\delta Q}{Q} \right)^2 = \left(\frac{\delta X}{X} \right)^2 + \left(\frac{\delta Y}{Y} \right)^2$$

This is why both mass and volume uncertainties contribute to the final density uncertainty.



KEY TAKEAWAY Density is a quotient of mass and volume. The relative uncertainty in density is the quadrature sum of the relative uncertainties in mass and volume. This directly follows from the general uncertainty propagation law.

$$\left(\frac{\delta \rho}{\rho} \right)^2 = \left(\frac{\delta m}{m} \right)^2 + \left(\frac{\delta V}{V} \right)^2$$

41. Which measurement dominates the uncertainty?

THE QUESTION

In the density calculation

$$\rho = \frac{m}{V}$$

both the mass m and the volume V have uncertainties.

Which one matters more?

RELATIVE CONTRIBUTIONS

From the relative-uncertainty formula

$$\left(\frac{\delta\rho}{\rho}\right)^2 = \left(\frac{\delta m}{m}\right)^2 + \left(\frac{\delta V}{V}\right)^2$$

the two terms on the right are the individual relative contributions to the uncertainty in ρ .

Mass contribution:

$$c_m = \frac{\delta m}{m}$$

Volume contribution:

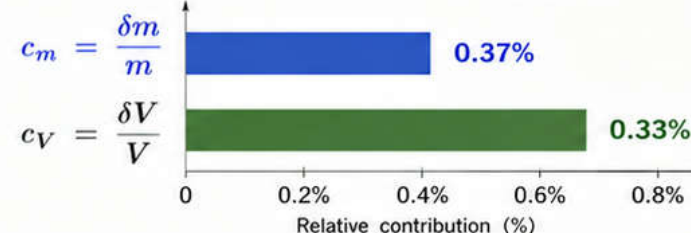
$$c_V = \frac{\delta V}{V}$$

COMPARE THE CONTRIBUTIONS

Use the relative values!

(dimensionless)

Typical example (ping-pong ball)



Here the volume uncertainty is *larger*, so it dominates the density uncertainty.

RULE OF THUMB

The larger relative contribution dominates the final uncertainty.

$$\text{If } c_1 \gg c_2, \text{ then } \delta\rho/\rho \approx c_1$$

In our example:

- $c_V \approx 0.33\%$ is larger than $c_m \approx 0.37\%$ ❌
- Therefore the volume measurement dominates. ✅
- The overall relative uncertainty is
 $\frac{\delta\rho}{\rho} = 0.50\% \approx 0.33\%$ (dominated by volume)

PRACTICAL CONSEQUENCE



Effort should focus on improving the dominant measurement.



Improving the less important measurement may hardly improve the final result.



Improving the dominant measurement reduces the final uncertainty.



GENERAL TAKEAWAY

Compare the two (or more) contributions:

$$c_i = \frac{\delta X_i}{X_i}$$

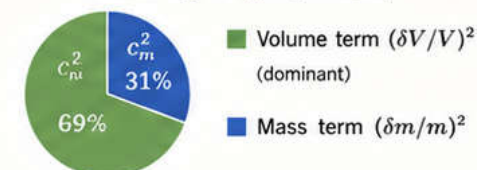
➡ The largest c_i dominates δQ .

➡ Improving the least important c_i may hardly improve δQ .

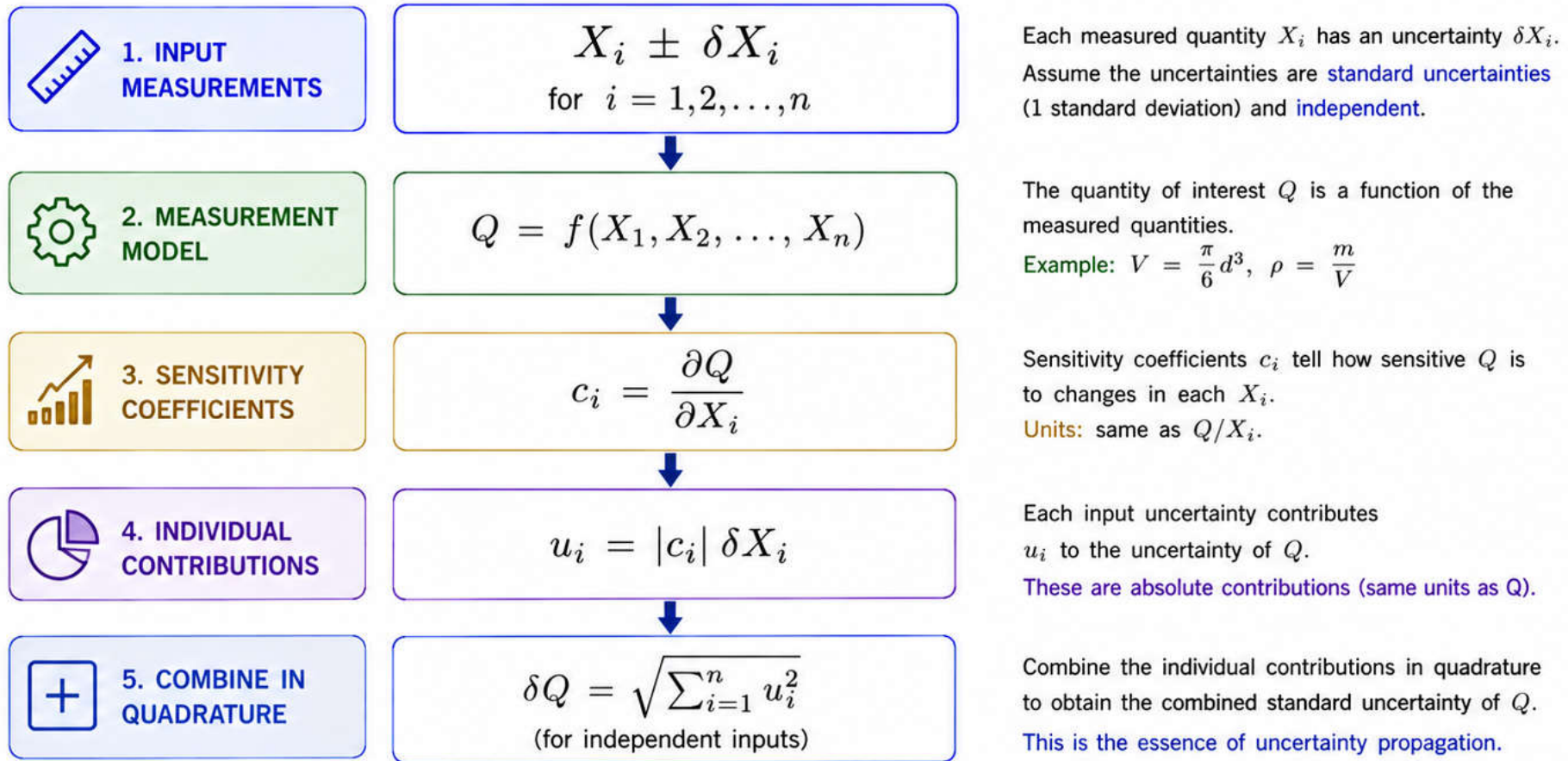


This is the essence of an uncertainty budget.

Uncertainty budget (example)



42. Uncertainty propagation: the complete picture



SUMMARY

$X_i \pm \delta X_i$
Measurements



$Q = f(X_1, X_2, \dots)$
Model



$c_i = \frac{\partial Q}{\partial X_i}$
Sensitivity



$u_i = |c_i| \delta X_i$
Contributions



$\delta Q = \sqrt{\sum_{i=1}^n u_i^2}$
Combined uncertainty

Compute, propagate,
and combine:
measure → model → result