Hughes planes and their collineation groups

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Projective Geometries

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Then, the quotient $\frac{V \setminus \{0\}}{\sim} = \mathrm{PG}(n, \mathbb{F})$ is called the n-dimensional projective space over $\mathbb{F}$.

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$\dots$

n-dimensional subspaces of $V \Rightarrow$ hyperplane (projective subspace of dimension $n-1$).

The finite case

If $\mathbb{F} = \mathrm{GF}(q)$ then we have the finite projective space of order q containing exactly

$$\frac{q^{n+1} - 1}{q - 1} = q^n + q^{n-1} + \cdots + q + 1$$

points.

Little Wedderburn's theorem

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The construction we have seen yields all possible finite projective geometries… when $n > 2$.

Projective planes

If $n = 2$ then we have a projective plane, that is, an incidence structure of points and lines satisfying:

- Every two distinct points are on exactly one line;
- Every two lines meet at exactly one point;
- There are four points no three of which are collinear.

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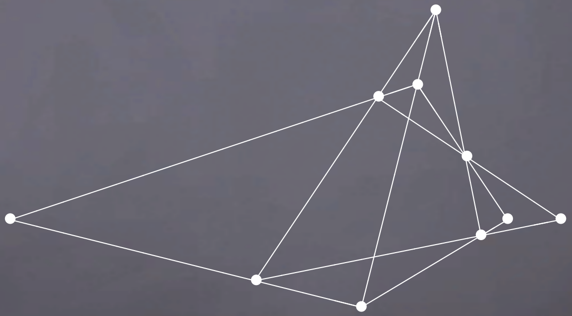
When $n = 2$ and $q \geq 9$ this construction does not produce all possible projective planes, but only the Desarguesian ones.

The Desargues' theorem



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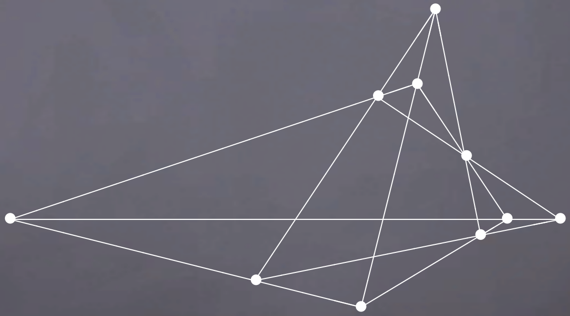

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- admit a collineation group of order 78, and
- be self-dual, that is isomorphic to its dual plane π^* which is obtained from π by interchanging the role of points and lines.

V-W systems and nearfields

A left (right) Veblen-Wedderburn system R is a finite set containing ar least two elements: 0 and 1, with two binary operations "+" and "$\cdot$" satisfying:

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Distributive laws:

$a(b + c) = ab + ac \Rightarrow$ left V-W system;
$(a + b)c = ac + bc \Rightarrow$ right V-W system.

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The centre of R is the set

$$Z(R) = \{\, z \in R \mid xz = zx \text{ for all } x \in R \,\}.$$

Exixtence of nearfields

Theorem (Zassenhous 1936)

There exists a nearfield R (which is not a field) of order $q^2 = p^{2h}$ *for any odd prime p and positive integer h whose centre is* $F \simeq \mathrm{GF}(q)$.

Exixtence of nearfields

Take $\mathrm{GF}(q^2)$ with q an odd prime power and define a new algebraic structure R with

elements: same as $\mathrm{GF}(q^2)$;

sum: same as $\mathrm{GF}(q^2)$;

multiplication:

$$x \circ y = \begin{cases} xy & \text{if } x \text{ is a square in } \mathrm{GF}(q^2) \\ xy^q & \text{if } x \text{ is not a square in } \mathrm{GF}(q^2). \end{cases}$$

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R is a nearfield (not a field) of order q^2 whose centre is $\mathrm{GF}(q)$.

Sporadic examples

Seven exceptional examples of orders 5^2, 11^2, 7^2, 23^2, again 11^2, 29^2 and 59^2 whose generators of the multiplicative group are represented by 2×2 matrices.

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For $q^2 = 11^2$, take the matrix group $\langle A, B, C \rangle$ generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 5 \\ -5 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix};$$

elements of R are of type $x = x_1 + x_2 t$ with $t^2 = \tau$, τ a non-square element of $\mathrm{GF}(q)$.

(Left) vector spaces

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V set of all triples (x, y, z) with $x, y, z \in R$, and V_0 subset of all triples (x, y, z) with $x, y, z \in F$.

Let A be a linear transformation of V_0 over F.

$$(x, y, z)A^m =$$
$$(a_{11}x + a_{12}y + a_{13}z,\ a_{21}x + a_{22}y + a_{23}z,\ a_{31}x + a_{32}y + a_{33}z),$$

where $a_{ij} \in F$ depend on m.

(Left) vector spaces

Suppose that A has the property that for any $\mathbf{v}_0 \in V_0$ $\mathbf{v}_0 A^m = k\mathbf{v}_0$ for some $k \in R \setminus \{0\}$ if and only if $m \equiv 0 \pmod{q^2 + q + 1}$.

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Then, A induces a linear transformation of V over R.

Further, V is partitioned into point-orbits of length q^2+q+1.

A new incidence structure π

Points

Triples $(x, y, z) \in V$ with the identification
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Lines

Formal symbols $L_t A^m$ where either $t = 1$ or $t \in R \setminus F$, $0 \leq m \leq q^2 + q$ and we write just L_t when $m = 0$.

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Incidence

The point (x, y, z) is on L_t if and only if $x + yt + z = 0$ while $L_t A^m$ contains all the points $(x, y, z)A^m$ such that (x, y, z) is in L_t.

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Theorem (Ryser 1950)

Let $X = \{x_1, \ldots, x_v\}$ be a finite set and let $T_1, \ldots, T_v$ be sets consistinf of elements from X. If each T_j contains exactly s distinct elements of X end every pair of distinct sets T_i, T_j shares exactly λ elements then

$$\lambda = \frac{s(s-1)}{v-1}.$$

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The lines of π_0 are those of the form $L_1 A^m$. There are exactly $q^2 + q + 1$ such lines.

A simpler construction

By Rosati (1960)

Back to R with multiplication defined by

$$x \circ y = \begin{cases} xy & \text{if } x \text{ is a square in } \mathrm{GF}(q^2) \\ xy^q & \text{if } x \text{ is not a square in } \mathrm{GF}(q^2). \end{cases}$$

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Let $\mathscr{P}$ and $\mathscr{L}$ be two copies of the set of triples (x, y, z) with $x, y, z \in R$ and identification as before.

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Let $P = (x, y, z) \in \mathscr{P}$ and $L = (u, v, w) \in \mathscr{L}$ with $u = a + a_1 t$, $v = b + b_1 t$, $w = c + c_1 t$, and $a, a_1, b, b_1, c, c_1 \in F$.

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Define a relation "$\sim$" between elements of $\mathscr{P}$ and elements of $\mathscr{L}$ where $P \sim L$ if and only if

$$xa + yb + zc + (xa_1 + yb_1 + zc_1) \circ t = 0.$$

A simpler construction

- The relation $\sim$ is independent of the choice of $t \in R \setminus F$.

- $P \sim L$ if and only if $L \sim P$.

- If $P \sim L$ and $k_1, k_2 \in R \setminus \{0\}$ then $(k_1 \circ P) \sim (k_2 \circ L)$.

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The realtion $\sim$ induces an incidence relation between $\mathscr{P}$ and $\mathscr{L}$, and the incidence structure $(\mathscr{P}, \mathscr{L}, \sim)$ so defined is a Hughes plane of order $q^2 = p^{2h}$.

A simpler construction

In this setting, for some $0 \leq m \leq q^2 + q$ the equation

$$xa + yb + zc + (xa_1 + yb_1 + zc_1) \circ t = 0$$

represents a line $L_t A^m$ in a natural manner.

Collineation groups

The Hughes plane π of order q^2 admits a cyclic collineation group S of order q^2+q+1, that is, some collineations of π can be obtained by extending to the whole plane the action of the collineations of the cyclic collineation group of its (Desarguesian) subplane π_0.

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For $q=3$ (Veblen-Wedderburn) π admits also a collineation group T of order 6 fixing π_0 pointwise. This is the automorphism group of the nearfield R which fixes $F=\mathrm{GF}(3)$ and is isomorphic to $\mathrm{Sym}(3)$.

Collineation groups

If $\theta \in \mathrm{Aut}(R) = T$, then under the action of θ we have that

$$L_1 A^m \mapsto L_1 A^m$$

for every $0 \leq m \leq q^2 + q$ while

$$L_t A^m \mapsto L_{t\theta} A^m$$

with $t \neq t\theta$ when $t \neq 1$.

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Is that it?

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hence $|\Sigma| = 6 \cdot 5{,}616 = 33{,}696$.

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If $q^2 = p^{2h}$ then T is a cyclic group of order $2h$.

If $q^2 = p^{2h} \neq 9$ then

$$|\Sigma| = 2mq^3(q^2 + q + 1)(q - 1)^2(q + 1),$$

that is,

$$|\Sigma| = 2|\mathrm{P\Gamma L}(3, q)|.$$

Collineation groups

In particular, when $q^2 = 25$ we have

$$|\Sigma| = 2 \cdot 31 \cdot 30 \cdot 25 \cdot 16 = 2 \cdot 372{,}000 = 744{,}000,$$

and Σ has two orbits on π:

- one orbit is π_0 and
- the other one is $\pi \setminus \pi_0$.

Another construction (Bose 1973)

In $\mathrm{PG}(2, q^2)$, with q an odd prime power, fix a real line as the line at the infinity ℓ_∞, let $\alpha = \mathrm{AG}(2, q^2)$ be the affine plane and $\alpha_0 = \mathrm{AG}(2, q)$ its real part.

There is a natural classification for the lines of α with respect to the collineation induced by the Frobenius automorphism $x \mapsto x^q$ of $\mathrm{GF}(q^2)$:

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- Lines of type 0_P: complex lines with real direction; ℓ^q is parallel to ℓ.
- Lines of type 1: complex lines with one real point and complex direction; $\ell^q \cap \ell = \{V\}$ with $V \in \alpha_0$ (the point V is the vertex of ℓ).

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Let ℓ be a line of type 1. Then, if $\mathbf{d} = (l, m)$ is the direction vector of ℓ, any point $P \in \ell$ can be written as

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$\mathcal{G}(\ell) = \{ P \in \ell \setminus \{V\} \mid P \text{ is green} \}$,

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  - the lines of type 0_p of $\mathrm{PG}(2, q^2)$ with the same "direction";
  - the lines defined as

$$\{V\} \cup \mathcal{R}(\ell) \cup \mathcal{G}(\ell^q)$$

with the "direction" of ℓ.

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The incidence structure so defined turns out to be a non-Desarguesian affine plane whose projective closure (by means of the line at the infinity ℓ_∞) is a Hughes plane of order q^2.

Concluding remarks

The Hughes plane π of order q^2, with q and odd prime power can be obtained from the Desarguesian plane $\mathrm{PG}(2, q^2)$ and one of its partitions into Baer subplanes.

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- Keep π_0 and give an alternative collinearity condition for the points on the complex lines (tangents to π_0).

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- The complex points in $\pi \setminus \pi_0$ are partitioned into $q^2 - q$ subsets Ω_i, with $|\Omega_i| = q^2 + q + 1$ for all $1 \leq i \leq q^2 - q$, coming from the subplanes other than π_0 in $\mathscr{P}$.

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- See de Resmini (1985) and (1987) for the combinatorial properties of these sets.