Seconda Università degli Studi di Napoli (a Caserta)
Dipartimento di Matematica e Fisica

Generalized Hyperfocused Arcs

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(jont work with Aart Blokhuis and Giuseppe Marino)

Finite Geometry Conference and Workshop
University of Szeged
10-14 June, 2013

Arcs and blocking sets

DEFINITION

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A line containing 1 or 2 points of a $k-$arc is said to be a tangent or secant to the $k-$arc, respectively.

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Let $\mathcal{F}$ be a set of lines in $PG(2,q)$. A blocking set of $\mathcal{F}$ is a set of points $\mathcal{B} \subset PG(2,q)$ having non-empty intersection with each line in $\mathcal{F}$. If this is the case, we also say that the lines in $\mathcal{F}$ are blocked by $\mathcal{B}$.

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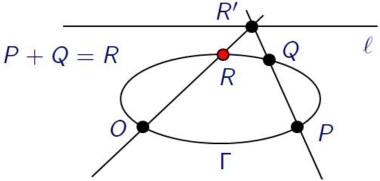

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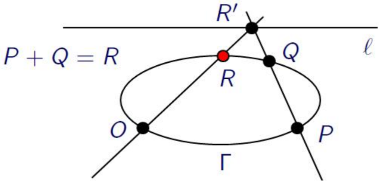

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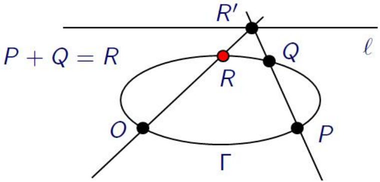

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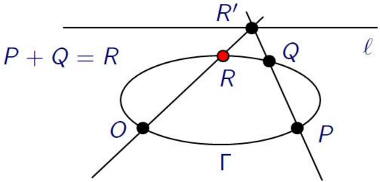

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- the sum of P and Q is defined by $P + Q = R$, where R is the second of the two points (counted with multiplicity) common to the line OR' and $\Gamma \setminus \ell$.

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A dual 3-net embedded in $PG(2, \mathbb{F})$*, is a triple* $\{A, B, C\}$ *with* A, B, C *pairwise disjoint point-sets of size n, called* components, *such that every line meeting two distinct components meets each component in precisely one point.*

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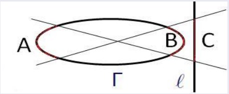

Now, given a proper subgroup A *of* G *of finite order* n *and one of its cosets* B $(\neq A)$, *the set* C *of the points of* ℓ *on some line intersecting both* A *and* B *has exactly* n *points.*

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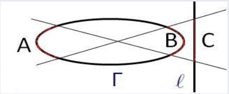

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The following theorem follows from the main result in the paper

Blokhuis A., Korchmáros G. & M.F. - 2011

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Theorem

Let $\{A, B, C\}$ be a dual 3-net of order n in $PG(2,\mathbb{F})$. Then, if C is contained in a line and $\mathbb{F}$ has positive characteristic $p \geq n$, $A \cup B$ is contained in a conic. If this conic is irreducible then it is of type described in the previous example.

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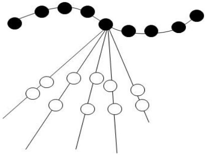

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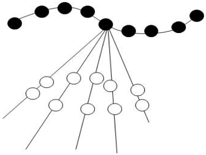

Moreover, the $k-1$ black points induce a factorization (i.e. a partition into matchings) of the white k-arc.

A trivial example of generalized hyperfocused arc

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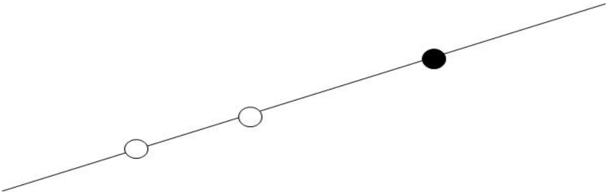
$\mathcal{B}$ consists of a unique black point out of $\mathcal{H}$ on the line through the two white points of $\mathcal{H}$.



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- It is known that there exist examples of generalized hyperfocused arcs which are not hyperfocused, provided q is even (M.Giulietti - E.Montanucci, 2006).
- The study of hyperfocused arcs is motivated by a relevant application to cryptography in connection with constructions of efficient secret sharing schemes (G.Simmons, 1990; L.Holder, 1997).

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These set of combinations is called an access structure and a solution to the problem is an example of a secret sharing scheme.

A geometric solution
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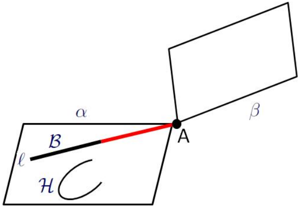

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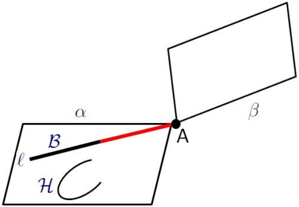

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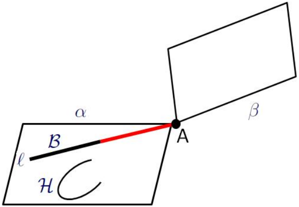

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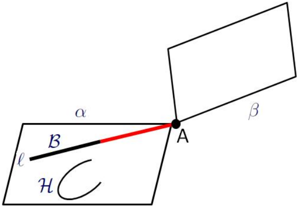

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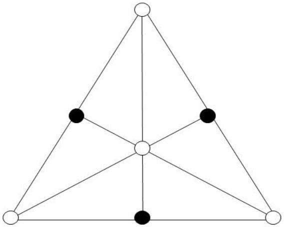
Then any team in the access structure can get the secret.

EXAMPLE

Any 4-arc $\mathcal{Q}$ of white points, with its three black diagonal points, is a non trivial example of generalized hyperfocused arc.



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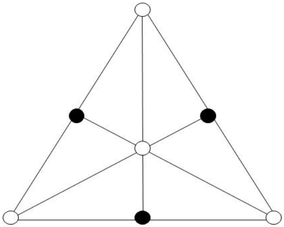

Note that:

- when q is even, the three black diagonal points of $\mathcal{Q}$ are collinear;
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A non trivial example of generalized hyperfocused arc
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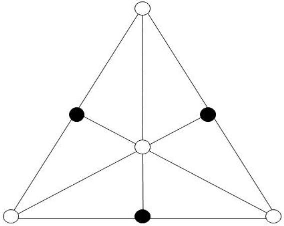

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THEOREM (A.Blokhuis - G.Marino - F.M., 2013)

The $4-$arc is the only non trivial example of generalized hyperfocused arc, provided q is an odd prime.

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- We write $A \triangleq B$ if $[A] = [B]$, i.e. $A = \lambda B$, for some non zero λ in $\mathbb{F}$.

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- For the following, let

$$\mathcal{H} = \{W_1 = [E_1], W_2 = [E_2], \ldots, W_{2n} = [E_{2n}]\}$$

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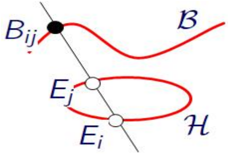
- We denote by B_{ij} the unique black point on the line $< W_i, W_j >$ and we define b_{ij} by



$$B_{ij} \triangleq E_i + b_{ij} E_j .$$

Since $B_{ji} = B_{ij}$ we have $b_{ji} = 1/b_{ij}$.

A basic relation
and a notation

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$$B_{ij} \stackrel{\triangle}{=} b_{1i}E_i + b_{1j}E_j, \text{ for all } \; i,j = 2, \dots, 2n.$$

  Taking into account that

$$B_{1j} \stackrel{\triangle}{=} E_1 + b_{1j}E_j \;\text{ and }\; B_{i1} \stackrel{\triangle}{=} E_i + b_{i1}E_1 \stackrel{\triangle}{=} E_1 + b_{1i}E_i \,,$$

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$$B_{ij} \stackrel{\triangle}{=} E_i + E_j \,, \text{ for all } \; i, j = 1, 2, \ldots, 2n.$$

Let ℓ be a line intersecting $\mathcal{B}$ in exactly $m < p$ points, set

$$S = \ell \cap \mathcal{B} = \{B_1, B_2, \ldots, B_m\}.$$

Intersection numbers of $\mathcal{B}$

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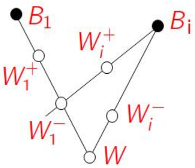

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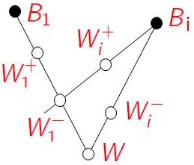

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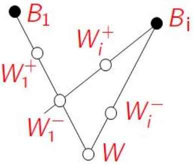

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Using appropriate white points, $\mathcal{H}$ can be partitioned into pairs of blocks, each block of size m, so m **must be a divisor of** n.

$S \cup S^- \cup S^+$ is a dual 3-net of order m

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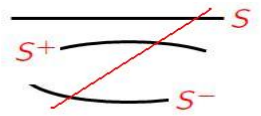

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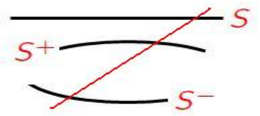

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By A.Blokhuis-G.Korchmáros-F.M. Theorem,

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The case $|\mathcal{H} \cap \ell| = 1$ cannot occur, otherwise m should divide p.

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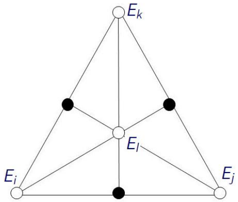
Using these two representations of S^+ we can prove:

Lemma

If a line ℓ intersects $\mathcal{B}$ in exactly $m<p$ points, then $m \leq 4$.

A second step



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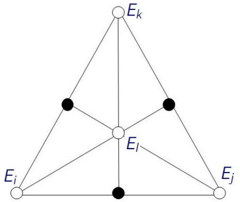

A 4-set $\{E_i, E_j, E_k, E_l\}$ of withe points of $\mathcal{H}$ is said to be *special* if $E_i + E_j + E_k + E_l = \mathbf{0}$. This means that $\{E_i, E_j, E_k, E_l\}$ is a non trivial generalized hyperfocused 4-arc contained in $\mathcal{H}$.

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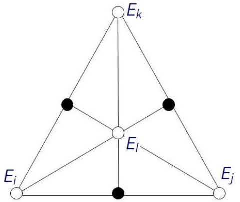

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Let $E_1, E_2, E_3, E_4, E_5, E_6$ be six distinct points of $\mathcal{H}$ such that

$$E_1 + E_2 \triangleq E_3 + E_4 \triangleq E_5 + E_6 \triangleq B\,.$$

Then, if $\{E_1, E_2, E_3, E_4\}$ and $\{E_1, E_2, E_5, E_6\}$ are special, $\{E_3, E_4, E_5, E_6\}$ is not.

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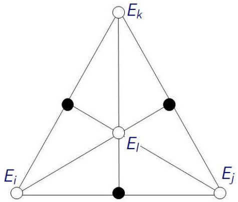

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This lemma ensures that, if $n > 2$, $\mathcal{H}$ contains a non special $4-$set of withe points.

The main result

Assume $n > 2$ and let E_1, E_2, E_3, E_4 be a non special $4-$set. Then we have 5 different black points:

$$B_{12} = B_{34},\ B_{13},\ B_{14},\ B_{23},\ B_{24}.$$

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Finally, using this frame and after long and non trivial calculations, we can prove our main result.

THEOREM

If p is an odd prime and $\mathcal{H}$ is a hyperfocused arc of size $2n$, in $PG(2,p)$ then $n \leq 2$.

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The strong cylinder conjeture (Ball, 2011)

A set C of q^2 points in $AG(3, q)$ that intersects every plane in 0 mod q points must be a cylinder, i.e. the union of q parallel lines.

The ordinary conjecture states the same thing for q a prime.

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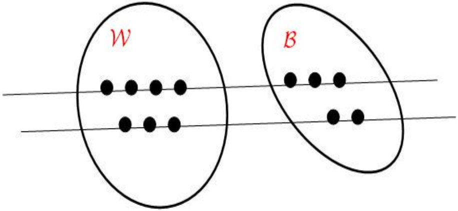
with the property that every line containing $m > 0$ white points contains exactly $m-1$ black points (counted with multiplicity).



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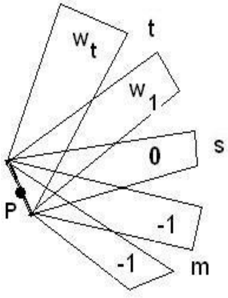
- Let the *weight* of a line α in π be $k - 1$, if α contains kq points of C as plane of $PG(3, q)$.
- All points of a line of weight -1 do not occur in the multiset X, that is they have weight 0 as points of X.



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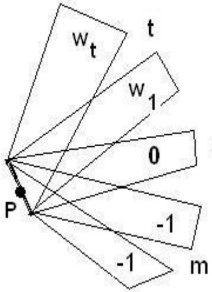

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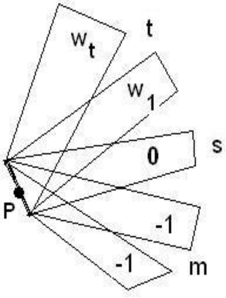

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- To classify these configurations in general seems to be hopeless but the prime case could be doable (and might settle the cylinder conjecture!).

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- **(White points form an arc)** The white points form a 4-arc, and there are 3 black points, the diagonal points.

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THANKS FOR ATTENTION