Diametric completions

Rolf Schneider

Universität Freiburg

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Let M be a nonempty bounded set in a metric space (X, ρ).

The diameter of M is

$$\operatorname{diam} M := \sup\{\rho(x, y) : x, y \in M\}.$$

M is diametrically complete if

$$\operatorname{diam}(M \cup \{x\}) > \operatorname{diam} M \quad \forall x \in X \setminus M.$$

A (diametric) completion of M is any diametrically complete set containing M and with the same diameter.

Every nonempty bounded set has a completion (many, in general); for example, by Zorn's lemma.

E. Akin: Maximal r-diameter sets and solids of constant width. arXiv:1003.5824v2

In Euclidean spaces, Meissner (1911) proved:

K is diametrically complete $\iff$ K is of constant width.

Recall that a convex body K in Euclidean space $\mathbb{R}^n$ is of constant width d if any two parallel supporting planes of K have distance d.

Equivalent:

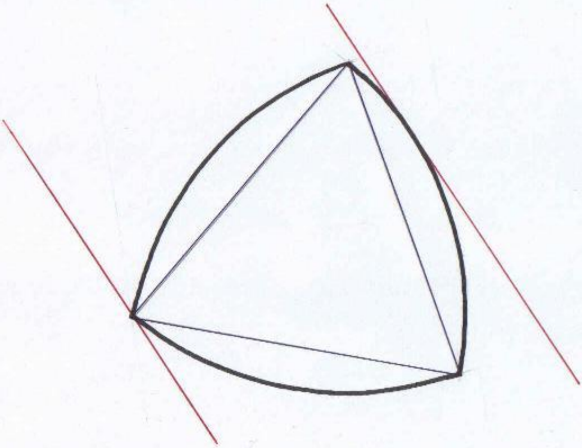
The support function of K satisfies

$$h(K,u) + h(K,-u) = d \quad \forall\, u.$$

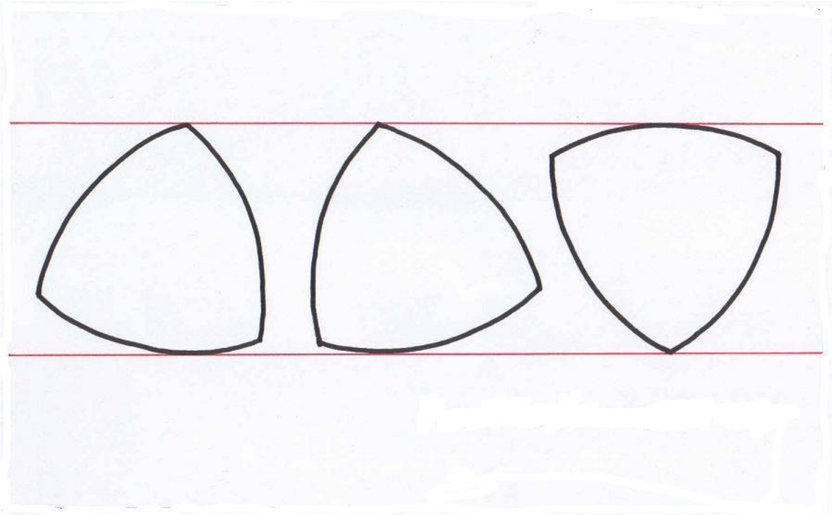
Equivalent:

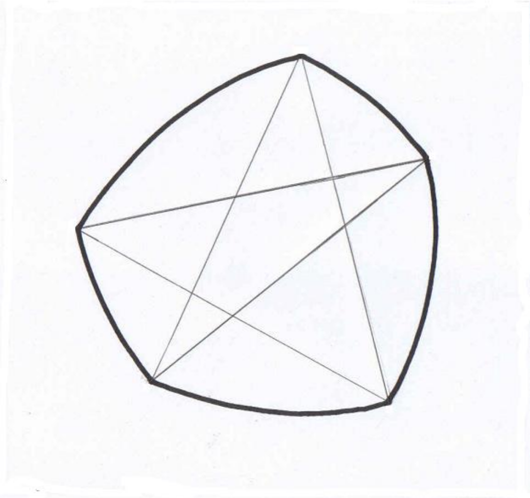
$$K + (-K) = B(o,d), \quad \text{ball of radius } d.$$

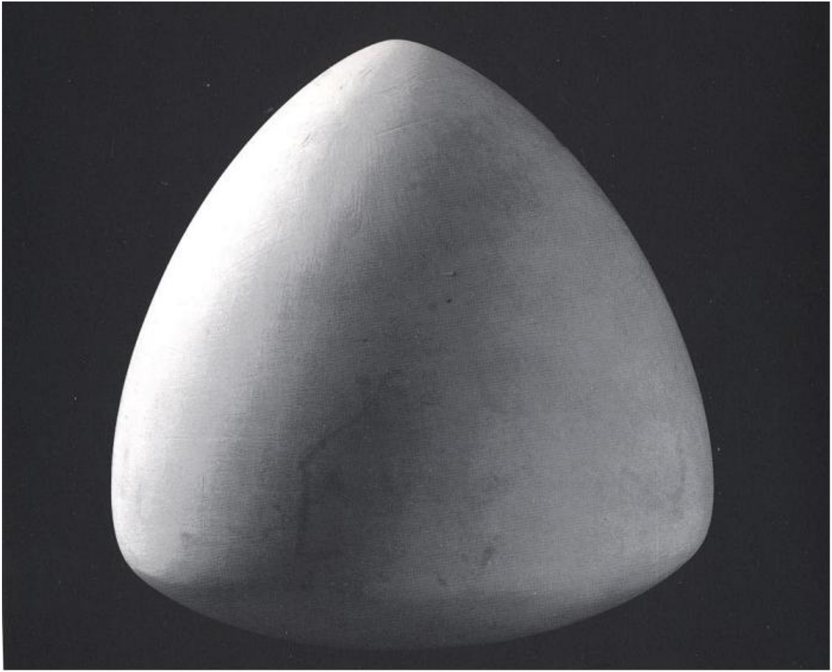
Reuleaux triangle



named after **Franz Reuleaux (1829–1905)**





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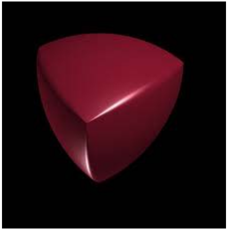

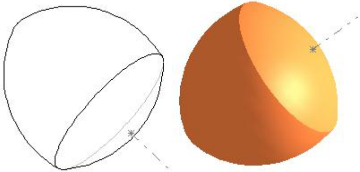

The space of bodies of constant width d in $\mathbb{R}^n$ is an infinite-dimensional closed convex set in $\mathcal{K}^n$, the space of convex bodies:

If K_1, K_2 are bodies of constant width d, then

$$(1-\lambda)K_1 + \lambda K_2 \qquad (0 \leq \lambda \leq 1)$$

is a body of constant width d.

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What about diametrically complete sets in Minkowski spaces?

Some answers are given in:

Joint work with José Pedro Moreno:

• Local Lipschitz continuity of the diametric completion mapping.
Houston J. Math.

• Diametrically complete sets in Minkowski spaces.
Israel J. Math.

• The structure of the space of diametrically complete sets in a Minkowski space.
Discrete Comput. Geom.

• Canonical diametric completions in Minkowski spaces.
(work in progress)

Minkowski spaces

$X = (\mathbb{R}^n, \|\cdot\|)$ Minkowski space:
a finite-dimensional real normed space

The norm $\|\cdot\|$ defines distance $\rho(x, y) := \|x - y\|$, width, diameter, unit ball $B := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, balls $\lambda B + z$.

Bodies of constant width and (diametrically) complete sets are defined as before.   Facts:

- Every set of diameter d is contained in a complete set of diameter d.

- K is of constant width $\implies$ K is diametrically complete

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- Every set of diameter d is contained in a complete set of diameter d.
- K is of constant width $\implies$ K is diametrically complete

Meissner stated the converse, and this was believed for more than 50 years (and 'reproved'), until Eggleston (1965) gave counterexamples. The situation is worse:

Theorem 1. *For a Minkowski space X, the following are equivalent:*

- *Every complete set is of constant width.*
- *The set of complete sets is convex.*
- *The set of completions of any given set is convex.*

Two-dimensional spaces have these properties.

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Theorem 2. (Yost 1991, M–S 2010)
Let $n \geq 3$. In the space of all n-dimensional Minkowski spaces, a dense open set of Minkowski spaces has the following properties:

- *The only bodies of constant width are balls.*
- *The sum of a complete body and a ball need not be complete.*
- *The set of completions of a given set need not be convex.*

Description of complete bodies

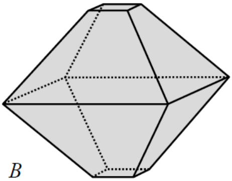
$K, M \in \mathcal{K}^n$, $\dim K = n$, $d > 0$

Supporting slab of K:
set bounded by two parallel supporting hyperplanes of K

A supporting slab of K is M-regular if at least one of the bounding hyperplanes of the parallel supporting slab of M contains a smooth boundary point of M.

Theorem 3. *K is a diametrically complete body of diameter d if and only if* (a) *and* (b) *hold:*

(a) *Every B-regular supporting slab of K has width $\leq d$.*
(b) *Every K-regular supporting slab of K has width $= d$.*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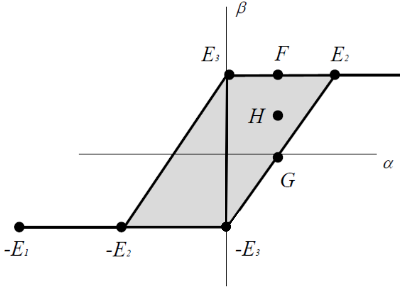
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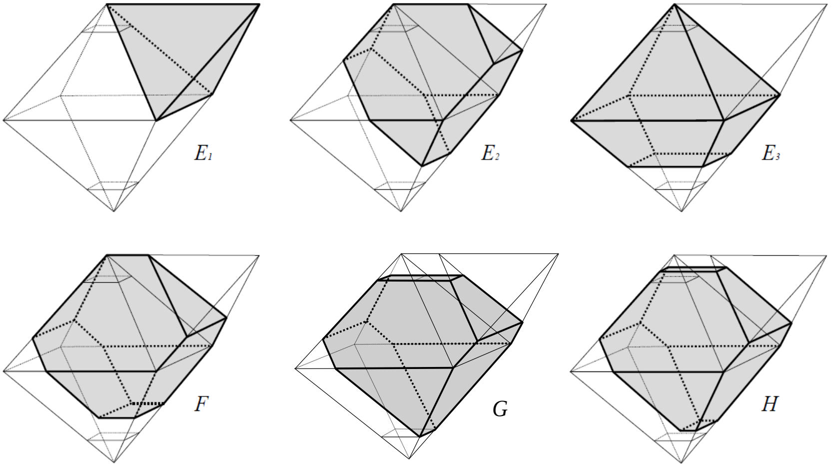
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On the space of diametrically complete sets

Let $\mathcal{D}_X$ be the space of translation classes of diametrically complete sets of diameter 2 in X.

Theorem 4. *If $X = (\mathbb{R}^n, \|\cdot\|)$ is polyhedral (i.e., the unit ball B is a polytope), then $\mathcal{D}_X$ is the union set of a finite polytopal complex.*

The proof uses representations of polyhedral sets introduced by McMullen (1973) (a variant of the Gale diagram technique).

Corollary. *If X is polyhedral, then $\mathcal{D}_X$ has only finitely many extreme points.*

Open problem: Does this characterize polyhedral norms? (Yes, if $n = 2$)

Let D_2 be the space of diametrically complete sets of diameter 2 in X.

In Euclidean space, D_2 is convex.

In a typical Minkowski space (in the sense of Baire category), D_2 is not even starshaped.

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A positive result:

Theorem 5. *The space D_2 is contractible.*

$\gamma(K) :=$ set of completions of K

$v_{\max}(K) := \max\{V(M) : M \in \gamma(K)\}$

$\gamma_{mv}(K) := \{M \in \gamma(K) : V(M) = v_{\max}(K)\}$

Groemer (1986): $\gamma_{mv}(K)$ consists of translates of one body

$\tau(K) := M - s(M)$ for any $M \in \gamma_{mv}(K)$, s Steiner point

loc. Lip. continuity of γ (M–S 2010) $\Rightarrow$ $v_{\max}$ is continuous

$\Rightarrow$ τ is continuous

For $K \in D_2$ and $\lambda \in [0,1]$, let $K_\lambda := (1-\lambda)K + \lambda B$ and

$$F(K,\lambda) := \tau\left(\frac{2}{\operatorname{diam} K_\lambda} K_\lambda\right) + (1-\lambda)s(K) + \lambda s(B).$$

Then $F : D_2 \times [0,1] \to D_2$ is continuous and $F(K,0) = K$, $F(K,1) = B$. Hence, D_2 is contractible.

Canonical completions

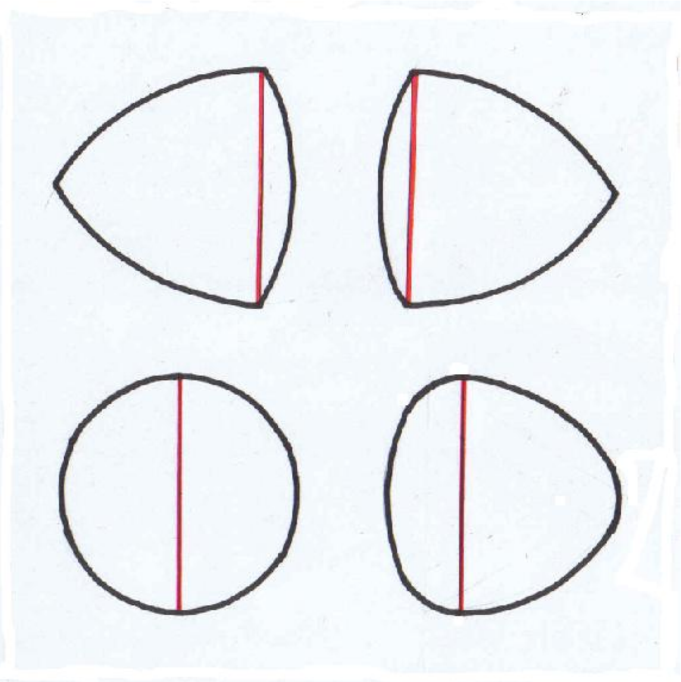
Is ‘completion’ continuous?

Let $\gamma(K)$ be the set of all completions of $K \in \mathcal{K}^n$.

Theorem 6. *The mapping* $\gamma : \mathcal{K}^n \to \mathcal{C}(\mathcal{K}^n)$ *is locally Lipschitz continuous, with respect to the Hausdorff metric* δ *induced by the norm on* $\mathcal{K}^n$ *and the Hausdorff metric* Δ *induced by* δ *on* $\mathcal{C}(\mathcal{K}^n)$*, the space of nonempty compact subsets of* $\mathcal{K}^n$*.*

In general, γ is many-valued.

For example, if K is a segment of length d in Euclidean space, then a suitable translate of any body of constant width d is a completion of K.



Does γ have a continuous selection?

The usual constructions of completions involve many arbitrary choices and hence cannot yield continuous completions.

This raises the question for ‘canonical’ completions.

A construction by Maehara (1984) can be slightly generalized.

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A construction by Maehara (1984) can be slightly generalized.

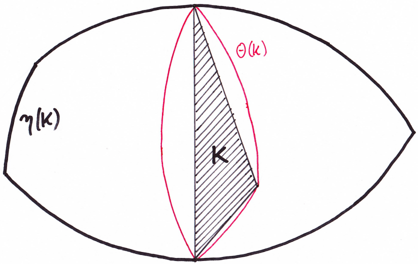
Definition. For $K \in \mathcal{K}^n$ of diameter d, let

$$\eta(K) := \bigcap_{x \in K} B(x, d), \qquad \theta(K) := \bigcap_{x \in \eta(K)} B(x, d).$$

Then

$$\mu(K) := \frac{1}{2}[\eta(K) + \theta(K)]$$

is the Maehara set of K.



Θ(K)
η(K)
K

Always true: $\mu(K)$ is a tight cover of K (i.e., contains K and has diameter d).

In Euclidean spaces: $\mu(K)$ is of constant width, and hence a completion of K.

Where does this work?

Definition. The norm $\|\cdot\|$ with unit ball B has the s-property if $B \cap (B + x)$ is a summand of B, for each x with $\|x\| \leq 1$.

Maehara (1984), Sallee (1987), Balashov & Polovinkin (2000), Karasëv (2001):

Theorem. *The Maehara set $\mu(K)$, for any $K \in \mathcal{K}^n$, is of constant width and hence a completion of K, if and only if the norm of X has the s-property.*

This completion is locally Lipschitz continuous:

Theorem 7. *Suppose that the norm of X has the s-property, and let $2C$ denote the Jung constant of X.*

Let $K, L \in \mathcal{K}^n$ be convex bodies with

$$\delta(K, L) \leq \epsilon \leq \frac{1}{3}(1 - C) \min\{d_K, d_L\}.$$

Then

$$\delta(\mu(K), \mu(L)) \leq \frac{7 - C}{1 - C}\,\epsilon \leq \frac{13}{2}(n + 1)\,\epsilon.$$

In particular, if X is a Euclidean space, then

$$\delta(\mu(K), \mu(L)) \leq 20\,\epsilon,$$

and if X is a two-dimensional Minkowski space, then

$$\delta(\mu(K), \mu(L)) \leq \frac{15}{2}\,\epsilon.$$

Two new properties of the Maehara completion in Euclidean spaces:

Recall that $\gamma(K)$ denotes the set of all completions of K.

Theorem 8. *The Maehara completion of K is a metric centre of $\gamma(K)$, that is, it minimizes the maximal Hausdorff distance from the elements of $\gamma(K)$.*

The Maehara completion of a convex body is at least as smooth as the body itself:

Theorem 9. *Every normal cone of the Maehara completion $\mu(K)$ is contained in some normal cone of K.*

Back to Minkowski spaces:

The only known examples of Minkowski spaces with the s-property are Euclidean spaces, two-dimensional Minkowski spaces, and $\oplus_\infty$ sums of such spaces.

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Therefore, for general Minkowski spaces, a different canonical completion procedure is needed.

For this, we extend a Euclidean method of Reinhardt (1922) ($n = 2$) and Bückner (1936) ($n = 3$).

In a first step, we replace each K by

$$\frac{1}{2}[\eta(K) + K],$$

which is a tight cover of K and has inradius at least $(2(n+1))^{-1}(\mathrm{diam}\, K)$.

The generalized Bückner completion

A convex body K of diameter d is complete if and only if

$$K = \bigcap_{x \in K} B(x, d)$$

(the spherical intersection property).

Aiming at completing a convex body K of diameter d, one is therefore tempted to consider

$$\eta(K) := \bigcap_{x \in K} B(x, d),$$

the wide spherical hull of K.

However, in general, $\operatorname{diam} \eta(K) > d$.

Bückner had the idea to consider a 'one-sided' version of the wide spherical hull, namely

$$C_u(K) := \eta(K) \cap Z^+(K,u) \quad \text{for given } u \neq o,$$

with

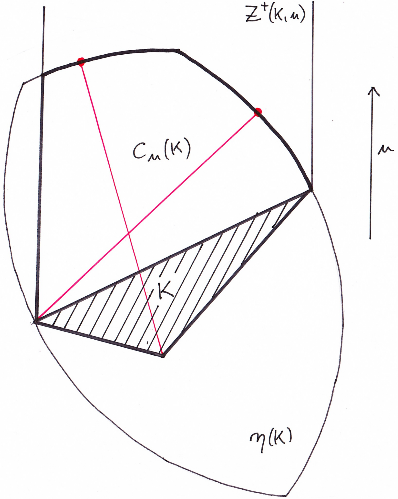
$$Z^+(K,u) := \{x + \lambda u : x \in K,\ \lambda \geq 0\}.$$

$C_u(K)$ **is a tight cover of** K!

But $C_u(K)$ is generally not complete.

However, it is 'partially complete':

Each 'upper' boundary point (w.r.t. u) is the endpoint of a diameter segment of $C_u(K)$.


$Z^+(K,u)$
u
$C_u(K)$
K
$\eta(K)$

Finitely many iterations, for m fixed directions $u_1, \ldots, u_m$ (where m depends only on n) yield a completion $C(K)$ of K.

We call C the generalized Bückner completion.

Theorem 10. *The generalized Bückner completion is locally Lipschitz continuous.*

Perfect norms

Recall:

Theorem 1. *For a Minkowski space X, the following are equivalent:*

- *Every complete set is of constant width.*
- *The set of complete sets is convex.*
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Two-dimensional spaces have these properties.

Definition. (Karasëv) A Minkowski space with these properties and its norm are called perfect.

Eggleston (1965) and Chakerian & Groemer (1983) have asked for a determination of all perfect Minkowski spaces.
This is still open.

We know from Maehara and Karasëv:

Theorem. *If the norm of X has the s-property, then X is perfect.*

Conjectures. Let K be a convex body of dimension ≥ 3 with the s-property. If K is either smooth (R. S. 1974) or strictly convex (Karasëv 2001), then it is an ellipsoid.

Theorem (Karasëv 2001). *A Minkowski space with a strictly convex norm is perfect if and only if its norm has the s-property.*

For general norms, this is not true.

Example:


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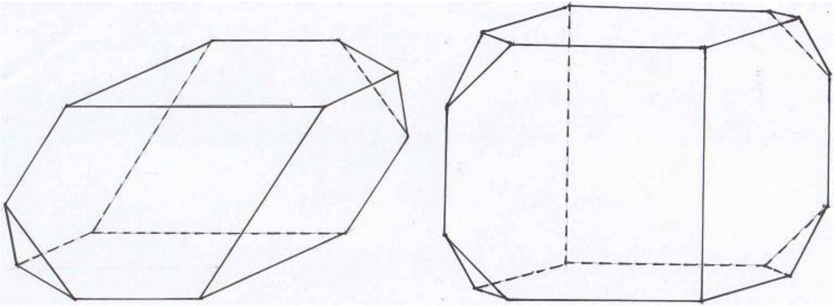


A new necessary condition:

Theorem 11. *If B is the unit ball of a perfect norm, then*

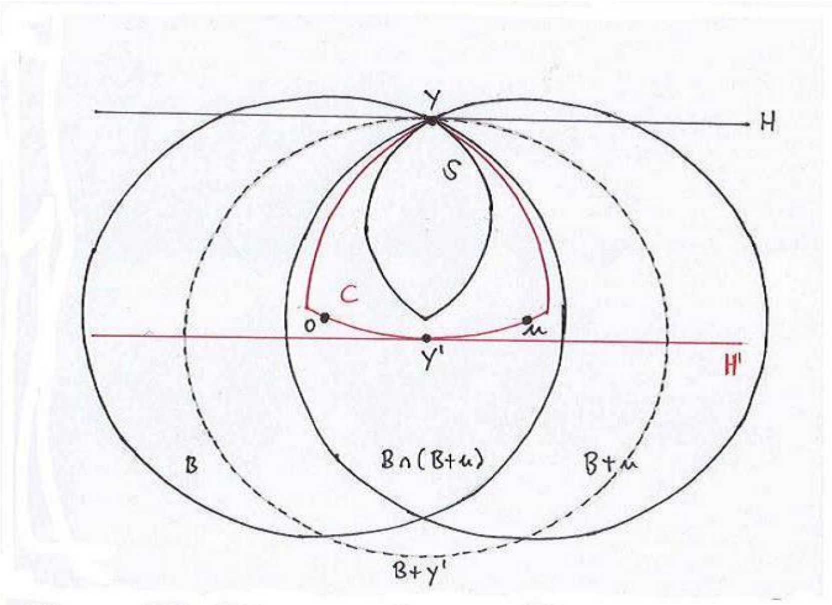
$$\frac{1}{2}\big(B \cap (B + x)\big)$$

is a summand of B for all x with $\|x\| \leq 1$.

Dürer's (1514) octahedron shows that the constant $\frac{1}{2}$ is best possible.

The proof of Theorem 11:

Lemma. *Let $K, L \in \mathcal{K}^n$. If to each supporting hyperplane H of K there are a point $x \in H \cap K$ and a vector $t \in \mathbb{R}^n$ such that $x \subset L + t \subset K$, then L is a summand of K.*



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B+Y'

The steps to prove Theorem 11:

$\|u\| \leq 1$, H arbitrary support plane to $B \cap (B+u)$

$y \in H \cap B \cap (B+u)$

$S := \frac{1}{2}((B \cap (B+u)) - y) + y \quad \Rightarrow \quad \operatorname{diam} S \cup \{o, u\} \leq 1$

$C :=$ completion of $S \cup \{o, u\} \quad \Rightarrow \quad C \subset B \cap (B+u)$

$\Rightarrow \quad H$ supports C at y

$H' :=$ support plane of C parallel to H

$\| \cdot \|$ is perfect $\quad \Rightarrow \quad \operatorname{dist}(H, H') = 1$

Let $y' \in C \cap H' \Rightarrow C \subset B + y'$

H supports $B + y'$, since $\operatorname{dist}(H, y') = 1$

Since H was arbitrary, the lemma shows that $\frac{1}{2}(B \cap (B+u))$ is a summand of B.

Thank you for your attention!

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And the organizers for the Workshop!