Gravitational waveforms for unequal mass black hole binaries

Márton Tápai,
Zoltán Keresztes , László Árpád Gergely
University of Szeged

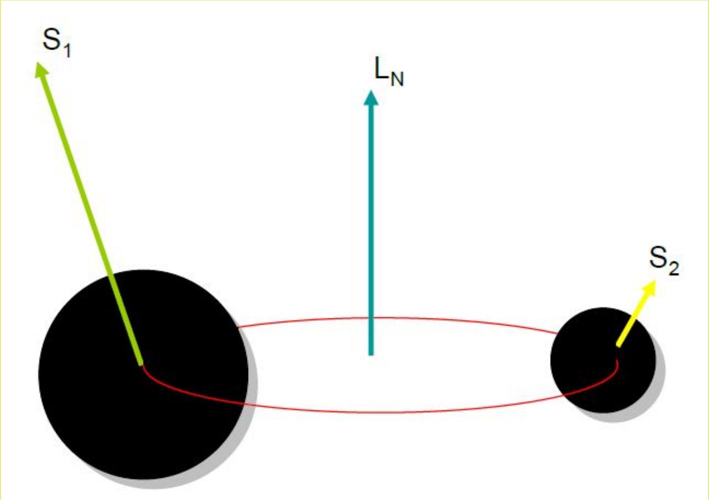
Outline

- Unequal mass black hole binaries
- Detection of gravitational waves
- Spin-dominated regime
- The gravitational waveform
- Limits of validity
- Phase of the gravitational waveform

Unequal mass binaries

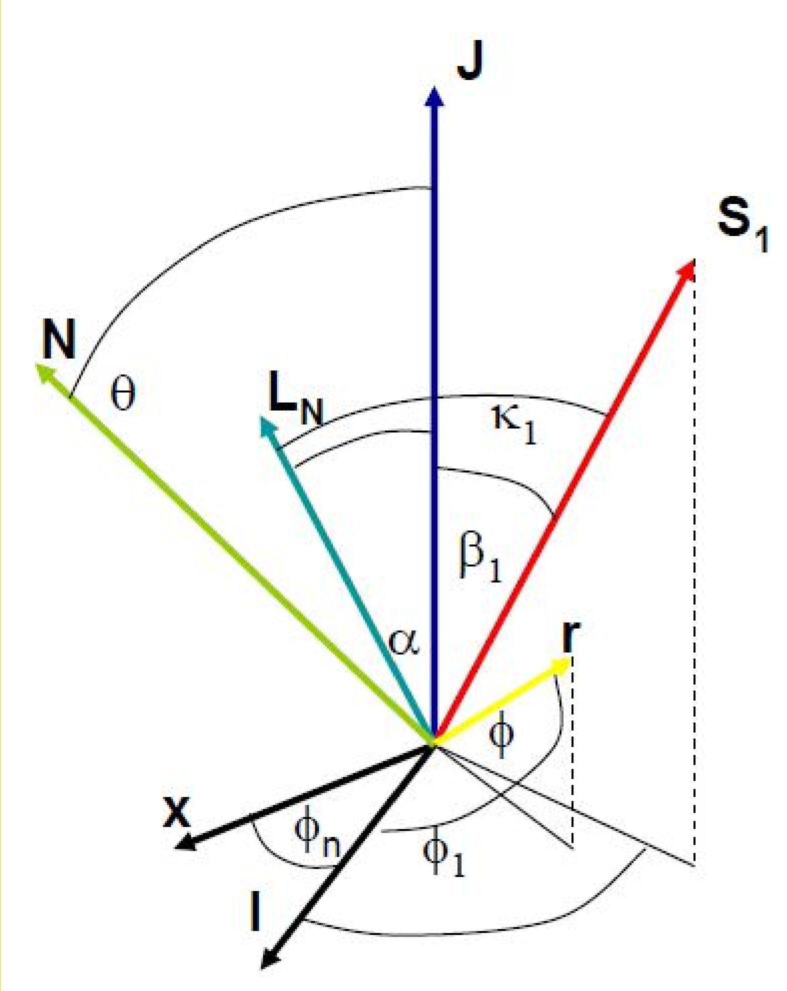
- Astrophysical black hole binaries
  - equal mass case not favored
- Supermassive black hole binaries
  - typical mass ratio: 0.3 ÷ 0.03

L. Á. Gergely, P. L. Biermann, Astrophys. J. 697, 1621 (2009).



Variables

- **N**: unit vector pointing to the observer from the source
- **r**: separation vector
- **l**: intersection of the planes perpendicular to **J** and $\mathbf{L_N}$
- x: arbitrary vector in the plane perpendicular to **J**



Search for gravitational waves

- Gravitational wave detectors
  - LIGO, Virgo, LISA, Einstein Telescope
- Search for waves with matched filtering
  - small SNR, template of waveforms needed
  - calculation time high
- Simple, but accurate waveforms are needed

Gravitational waveforms

- Post-Newtonian (PN) gravitational waveforms were previously calculated

L. E. Kidder, Phys. Rev. D 52, 821 (1995).

K. G. Arun, A. Buonanno, G. Faye, E. Ochsner, Phys. Rev. D 79, 104023 (2009).

- Approximate waveforms for equal mass case by Arun et al.
- Our aim is to get an approximation for unequal mass binaries

Spin-Dominated regime

- Ratio of spins $\frac{S_2}{S_1} = \frac{\chi_2}{\chi_1}\nu^2$

$$S_i \equiv \frac{G}{c} m_i^2 \chi_i$$

- Small mass ratios (ν)

$S_2 << S_1$

- S_2 neglected
- Ratio of the Newtonian orbital angular momentum (L_N) and larger spin S_1:

$$\frac{S_1}{L_N} \approx \varepsilon^{1/2} \nu^{-1} \chi_1$$

Spin-Dominated regime

- The PN parameter $\varepsilon = \frac{Gm}{c^2 r} \approx \left(\frac{v}{c}\right)^2$ increases as the black holes approach each other
- Small ν

$$\frac{S_1}{L_N} \approx \varepsilon^{1/2} \nu^{-1} \chi_1$$

- At the end of the inspiral S_1 dominates over L_N

Spin-Dominated regime

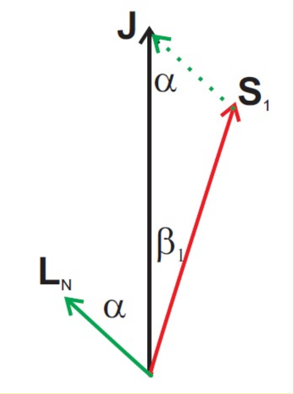
- $S_1 > L_N \longrightarrow$ introduce new small parameter

$$\xi = \varepsilon^{-1/2}\nu \leq 0.1$$

- Keeping terms up to $\varepsilon^{1.5}$, and neglect ξ^2
- Total angular momentum (J) conserved to 2 PN order

L. E. Kidder, C. M. Will, A. G. Wiseman, Phys. Rev. D 47, R4183 (1993).

$$\sin\beta_1 = \left(1+\frac{7}{2}\varepsilon\right)\frac{L_N}{S_1}\sin\alpha = \left(1+\frac{7}{2}\varepsilon\right)\frac{\xi}{\chi_1}\sin\alpha$$



- Angle span by J and S_1 (β_1) is small, of order ξ

Spin-Dominated Waveforms (SDW)

- Double expansion in the small parameters ε and ξ
- Structure of the waveforms:

$$h_{\substack{+\\ \times}} = \frac{2G^2 m^2 \varepsilon^{1/2} \xi}{c^4 D r} \left\{ h^{0}_{\substack{+\\ \times}} + \beta_1 h^{0\beta}_{\substack{+\\ \times}} + \varepsilon^{1/2} \left(h^{0.5}_{\substack{+\\ \times}} + \beta_1 h^{0.5\beta}_{\substack{+\\ \times}} - 2\xi h^{0}_{\substack{+\\ \times}} \right) + \varepsilon \left(h^{1}_{\substack{+\\ \times}} + \xi \left[h^{1,\xi}_{\substack{+\\ \times}} - 2 h^{0.5}_{\substack{+\\ \times}} \right] + \beta_1 h^{1\beta}_{\substack{+\\ \times}} \right.\right.$$
$$\left.\left. + h^{1SO}_{\substack{+\\ \times}} + \beta_1 h^{1\beta SO}_{\substack{+\\ \times}} \right) + \varepsilon^{3/2} \left(h^{1.5}_{\substack{+\\ \times}} + h^{1.5SO}_{\substack{+\\ \times}} + h^{1.5tail}_{\substack{+\\ \times}} \right) \right\}$$

Structure of SDW

terms from double expansion through β_1

terms from double expansion through ξ

$$h_{\substack{+\\ \times}} = \frac{2G^2 m^2 \varepsilon^{1/2}\xi}{c^4 D r}\left\{h^{0}_{\substack{+\\ \times}} + \beta_1 h^{0\beta}_{\substack{+\\ \times}} + \varepsilon^{1/2}\left(h^{0.5}_{\substack{+\\ \times}} + \beta_1 h^{0.5\beta}_{\substack{+\\ \times}} - 2\xi h^{0}_{\substack{+\\ \times}}\right) + \varepsilon\left(h^{1}_{\substack{+\\ \times}} + \xi\left[h^{1,\xi}_{\substack{+\\ \times}} - 2h^{0.5}_{\substack{+\\ \times}}\right] + \beta_1 h^{1\beta}_{\substack{+\\ \times}}\right.\right.$$
$$\left.\left. + h^{1SO}_{\substack{+\\ \times}} + \beta_1 h^{1\beta SO}_{\substack{+\\ \times}}\right) + \varepsilon^{3/2}\left(h^{1.5}_{\substack{+\\ \times}} + h^{1.5SO}_{\substack{+\\ \times}} + h^{1.5tail}_{\substack{+\\ \times}}\right)\right\}$$

spin-orbit contributions

gravitational wave tail

Leading order terms

$$
\begin{aligned}
4h_+^0 &= \sum_{+,-}\left[k_3\cos(2\phi_n \pm 2\psi)c_1^{(\pm 0)} - 2\sin\kappa_1 \sin 2\theta \sin(\phi_n \pm 2\psi)k^{(\pm)}\right] + 6\sin^2\kappa_1 \sin^2\theta\cos 2\psi \\
2h_\times^0 &= \cos\theta \sum_{+,-}\left[\sin(2\phi_n \pm 2\psi)c_1^{(\pm 0)} - 2\sin\theta\sin\kappa_1\cos(\phi_n \pm 2\psi)k^{(\pm)}\right] \\
2h_+^{0\beta} &= \sum_{+,-}\left[\sin 2\theta \sin(\phi_n \pm 2\psi)c_2^{(\pm 0)} + \sin\kappa_1 k_3 \sum_i \cos(2\phi_n \pm 2\psi)k^{(\pm)}\right] - 3\sin 2\kappa_1 \sin^2\theta\cos 2\psi \\
h_\times^{0\beta} &= \sum_{+,-}\left[\cos\theta\sin\kappa_1\sin(2\phi_n \pm 2\psi)k^{(\pm)} + \sin\theta\cos(\phi_n \pm 2\psi)c_2^{(\pm 0)}\right]
\end{aligned}
$$

- coefficients defined as

$$
c_{(i)}^{(\pm n)} = a_{(i)}^{(\pm n)} + \sin^2\kappa_1 \sum_{j=0}^{1}\left(b_{(i)j}^{(\pm n)} + d_{(i)j}^{(\pm n)}\cos\kappa_i\right)\sin^{2j}\kappa_1
$$

$$
\begin{aligned}
k^{(-)} &= \cos\kappa_1 + 1 \\
k^{(+)} &= \cos\kappa_1 - 1 \\
k_3 &= \sin^2\theta - 2 \\
k_4 &= 2 - 3\sin^2\theta
\end{aligned}
$$

Non-precessing case

- In this case $\kappa_1 = 0$ or $\kappa_1 = \pi$
- Only the coefficient a remains, with $k^+ = 0$ and $k^- = 2$

n	i	$a_{(i)}^{(\pm n)}$
0	1	$\mp 2k^{(\pm)}$
	2	$\mp k^{(\pm)}$
0.5	1	$4k^{(\pm)}\left(6-\sin^2\theta\right)$
	2	$4k^{(\pm)}$
	3	$\pm 2\left(6-\sin^2\theta\right)k^{(\pm)}$
	4	$12k^{(\pm)}$
	5	$\pm 2k^{(\pm)}(2\sin^2\theta-3)$
	6	$-2k^{(\pm)}$
	7	$\mp 2c_1^{\beta(\pm)}\left(6-\sin^2\theta\right)$
	8	$\pm 2k^{(\pm)}$
	9	$44-34\sin^2\theta \pm 2\left(5\sin^2\theta-46\right)\cos\kappa_1$
	10	$-2k^{(\pm)}\left(3-2\sin^2\theta\right)$
1	1	$\pm 8k^{(\pm)}$
	2	$6k^{(\pm)}\left(\sin^2\theta+5\right)$
	3	$2k^{(\pm)}\left(4-\sin^2\theta\right)$
	4	$\pm 2k^{(\pm)}\left(2\sin^4\theta+11\sin^2\theta-38\right)$
	5	$6k^{(\pm)}\left(3\sin^2\theta+5\right)$
	6	$\pm 2k^{(\pm)}\left(4\sin^2\theta+19\right)$
	7	$-2k^{(\pm)}\left(3\sin^2\theta-4\right)$
	8	$\pm 2k^{(\pm)}\left(4-\sin^2\theta\right)$
	9	$\pm 6k^{(\pm)}\left(5+\sin^2\theta\right)$
	10	$\mp 4k^{(\pm)}$
	11	$k^{(\pm)}\left(22+29\sin^2\theta-16\sin^4\theta\right)$
	12	$2k^{(\pm)}$
	13	$\pm 6k^{(\pm)}\left(3\sin^2\theta+5\right)$
	14	$-k^{(\pm)}\left(20\sin^2\theta+11\right)$
	15	$\mp 2k^{(\pm)}\left(3\sin^2\theta-4\right)$

n	i	$a_{(i)}^{(\pm n)}$
1.5	1	$\pm 4k^{(\pm)}\left(\sin^2\theta-6\right)$
	2	$\pm 4k^{(\pm)}\left(\sin^4\theta+42\sin^2\theta-166\right)$
	3	$16k^{(\pm)}$
	4	$8k^{(\pm)}\left(\sin^4\theta+8\sin^2\theta-28\right)$
	5	$8k^{(\pm)}\left(-332+94\sin^2\theta+\sin^4\theta\right)$
	6	$\pm 8k^{(\pm)}\left(38-42\sin^2\theta-9\sin^4\theta\right)$
	7	$-16k^{(\pm)}\left(152-46\sin^2\theta-9\sin^4\theta\right)$
	8	$\pm 24k^{(\pm)}\left(3\sin^2\theta-10\right)$
	9	$-8k^{(\pm)}\left(160-204\sin^2\theta-63\sin^4\theta\right)$
	10	$\pm 4k^{(\pm)}\left(3-2\sin^2\theta\right)$
	11	$-8k^{(\pm)}\left(14+3\sin^2\theta\right)$
	12	$-16k^{(\pm)}\left(15\sin^2\theta+76\right)$
	13	$-8k^{(\pm)}\left(5\sin^2\theta+166\right)$
	14	$-8k^{(\pm)}\left(80+63\sin^2\theta\right)$
	15	$\pm 4k^{(\pm)}\left(166-125\sin^2\theta-8\sin^4\theta\right)$
	16	$\mp 8k^{(\pm)}\left(38-61\sin^2\theta-24\sin^4\theta\right)$
	17	$\pm 8k^{(\pm)}\left(5-4\sin^2\theta\right)$

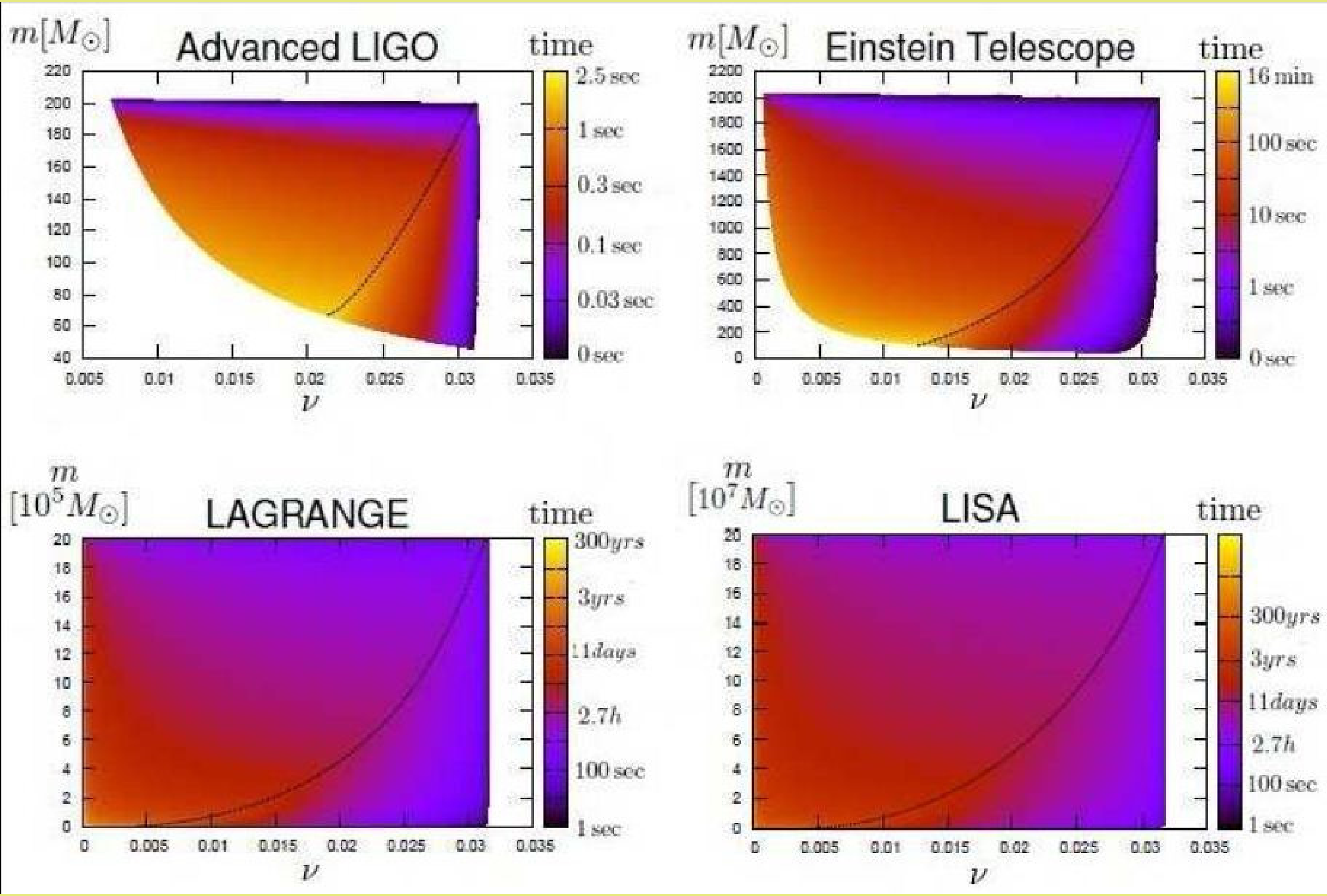
Limits of validity

- Our approximation holds
  - From $\xi = \varepsilon^{-1/2}\nu \leq 0.1 \longrightarrow \varepsilon_1 = Gm/c^2 r_1 = 100\nu^2$
  - To $\varepsilon_2 = 0.1$

J. Levin, S. T. McWilliams, H. Contreras, Class. Quant. Grav. 28 175001 (2011).

- $[\varepsilon_1, \varepsilon_2]\ \ exists\ if\ \nu < \nu_{\max} = 0.316 = 1 : 32$
- For how long (Δt) is the SDW in the sensitivity range of detectors?

$$\Delta t = \frac{Gm}{2^8 c^3} \frac{(1+\nu)^2}{5\nu} \left(\varepsilon_1^{-4} - \varepsilon_2^{-4}\right)$$



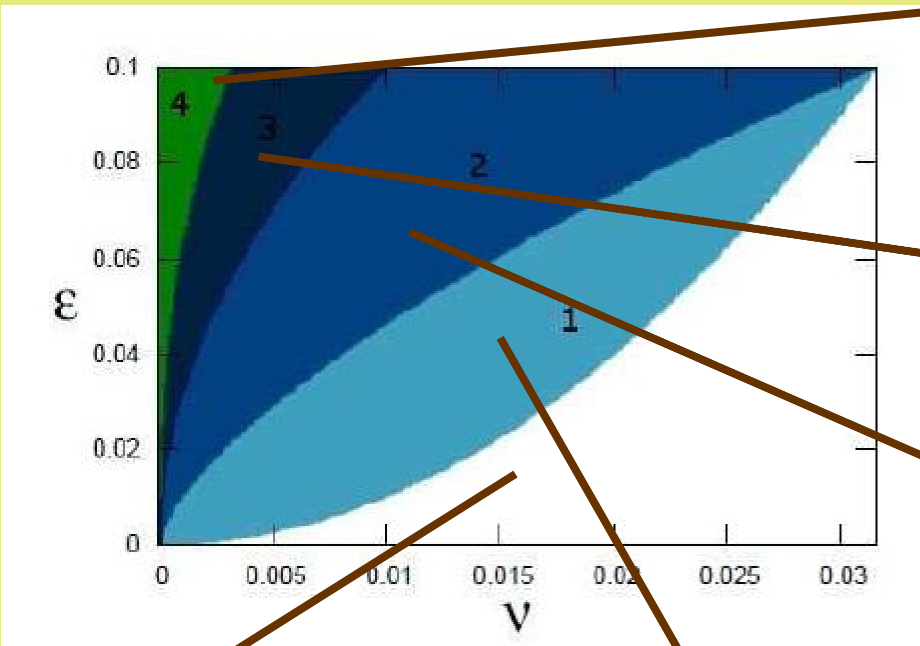
$m[M_\odot]$
Advanced LIGO
time
2.5 sec
1 sec
0.3 sec
0.1 sec
0.03 sec
0 sec
220
200
180
160
140
120
100
80
60
40
0.005
0.01
0.015
0.02
0.025
0.03
0.035
ν
$m[M_\odot]$
Einstein Telescope
time
16 min
100 sec
10 sec
1 sec
0 sec
2200
2000
1800
1600
1400
1200
1000
800
600
400
200
0
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
ν
m $[10^5 M_\odot]$
LAGRANGE
time
$300yrs$
$3yrs$
$11days$
$2.7h$
100 sec
1 sec
20
18
16
14
12
10
8
6
4
2
0
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
ν
m $[10^7 M_\odot]$
LISA
time
$300yrs$
$3yrs$
$11days$
$2.7h$
100 sec
1 sec
20
18
16
14
12
10
8
6
4
2
0
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
ν

Parameter evolution

- As ε increases throughout the inspiral
  - β_1 doesn't
  - $\xi = \varepsilon^{-1/2}\nu \leq 0.1$ does
- However ε^2 increases at a faster rate

- What terms do we need to keep as ε^2 increases?

What terms to keep?



$h^{0}_{\substack{+\\ \times}}\ h^{0.5}_{\substack{+\\ \times}}\ h^{1}_{\substack{+\\ \times}}\ h^{1SO}_{\substack{+\\ \times}}\ h^{1.5}_{\substack{+\\ \times}}\ h^{1.5SO}_{\substack{+\\ \times}}\ h^{1.5tail}_{\substack{+\\ \times}}$

$h^{0}_{\substack{+\\ \times}}\ h^{0\beta}_{\substack{+\\ \times}}\ h^{0.5}_{\substack{+\\ \times}}\ h^{1}_{\substack{+\\ \times}}\ h^{1SO}_{\substack{+\\ \times}}\ h^{1.5}_{\substack{+\\ \times}}\ h^{1.5SO}_{\substack{+\\ \times}}\ h^{1.5tail}_{\substack{+\\ \times}}$

$h^{0}_{\substack{+\\ \times}}\ h^{0\beta}_{\substack{+\\ \times}}\ h^{0.5}_{\substack{+\\ \times}}\ h^{0.5\beta}_{\substack{+\\ \times}}\ h^{1}_{\substack{+\\ \times}}\ h^{1SO}_{\substack{+\\ \times}}\ h^{1.5}_{\substack{+\\ \times}}\ h^{1.5SO}_{\substack{+\\ \times}}\ h^{1.5tail}_{\substack{+\\ \times}}$

$h^{0}_{\substack{+\\ \times}}\ h^{0\beta}_{\substack{+\\ \times}}\ h^{0.5}_{\substack{+\\ \times}}\ h^{0.5\beta}_{\substack{+\\ \times}}\ h^{1}_{\substack{+\\ \times}}\ h^{1\xi}_{\substack{+\\ \times}}\ h^{1\beta}_{\substack{+\\ \times}}\ h^{1SO}_{\substack{+\\ \times}}\ h^{1\beta SO}_{\substack{+\\ \times}}\ h^{1.5}_{\substack{+\\ \times}}\ h^{1.5SO}_{\substack{+\\ \times}}\ h^{1.5tail}_{\substack{+\\ \times}}$

The SDW is not valid in this region

Phase of the gravitational wave

- Orbital angular frequency evolution up to 2 PN order (B. Mikóczi, M. Vasúth, L. Á. Gergely, Phys. Rev. D 71, 124043 (2005).)

$$
\begin{aligned}
\frac{d\omega}{dt} &= \frac{96\,(Gm)^{5/3}\,\eta\omega^{11/3}}{5c^5}\left[1-\left(\frac{743}{336}+\frac{11}{4}\eta\right)\left(\frac{Gm\omega}{c^3}\right)^{2/3}+(4\pi-\beta)\left(\frac{Gm\omega}{c^3}\right)\right.\\
&\quad\left.+\left(\frac{34103}{18144}+\frac{13661}{2016}\eta+\frac{59}{18}\eta^2+\sigma\right)\left(\frac{Gm\omega}{c^3}\right)^{4/3}\right]\\
\beta &= \frac{1}{12}\left[\sum_{i=1}^{2}\chi_i\cos\kappa_i\left(\frac{113\nu^{2(i-1)}}{(1+\nu)^2}+75\eta\right)\right]\\
\sigma &= \sigma_{S_1S_2}+\sigma_{SS-self}+\sigma_{QM}
\end{aligned}
$$

- Integrating twice gives the phase
- After the double expansion:

$$
\phi_c-\phi = \frac{\varepsilon^{-3}}{32\xi}\left\{1+2\varepsilon^{1/2}\xi+\frac{1195}{1008}\varepsilon+\left(-10\pi+\frac{3925}{504}\xi+\frac{175}{8}\chi_1\cos\kappa_1\right)\varepsilon^{3/2}+\left[-\frac{21\,440\,675}{1016\,064}+\chi_1^2\left(\frac{375}{16}-\frac{3425}{96}\sin^2\kappa_1\right)\right]\varepsilon^2\right\}
$$

Summary

- Derived a waveform based on
  - small mass ratio $\longrightarrow$ ν^2 neglected
  - considering the last part of the inspiral
- Introduced a small parameter ξ, and double expanded the waveforms in ε and ξ
- Examined the validity of SDW
- Gave the phase in this approximation

Thank you for your attention

Tail term

- The gravitational wave tail from 1.5 PN amplitude correction gives some contributions that can be observed into the phase by redefine it as:

$$\psi = \phi - 2\varepsilon^{3/2} \ln\left(\frac{\omega}{\omega_0}\right)$$

Acknowledgement

- This presentation was supported by the European Union and co-funded by the European Social Fund. Project number: TÁMOP-4.2.2/B-10/1-2010-0012
- Project title: “Broadening the knowledge base and supporting the long term professional sustainability of the Research University Centre of Excellence at the University of Szeged by ensuring the rising generation of excellent scientists.”

National Development Agency
www.ujszechenyiterv.gov.hu
06 40 638 638
HUNGARY'S RENEWAL
The project is supported by the European Union and co-financed by the European Social Fund.

ÚJ SZÉCHENYI TERV