Projective realization of (finite) groups

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Overview

1. Configurations in the projective plane
2. Examples of dual 3-nets
3. Projective realization of finite groups
4. Dual 4-nets

Notations

- Let G be a finite group, $n = |G|$.
- Let K be a field of characteristic p such that $p = 0$ or $p > n$.
- We work in the projective plane $PG(2, K)$ over K, that is,
- points are homogeous triples (x, y, z) with $x, y, z \in K$, and
- lines are given by homogenous linear equations $aX + bY + cZ = 0$ with $a, b, c \in K$.
- Two objects are projectively equivalent if one can be transformed into the other by a projective linear transformation.
- The principe of duality says that the role of points and lines of a projective plane can be interchanged.

Sylvester-Gallai configurations

Sylvester-Gallai theorem

Let X be a finite set of points in the real projective plane without 2-secants. Then X is contained in a line.

Proof. See the Book. □

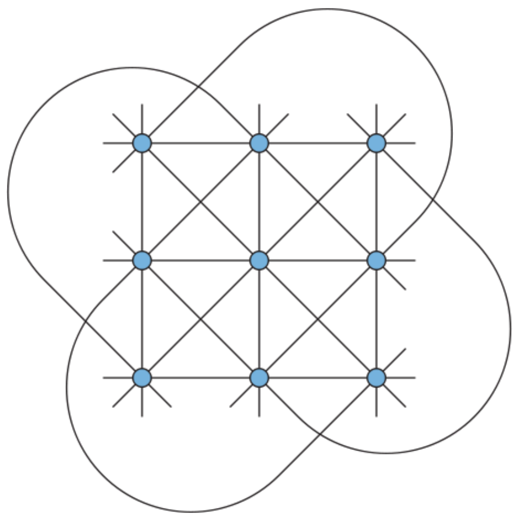
Definition: Sylvester-Gallai configurations

A finite set of points without 2-secants is called a Sylvester-Gallai configuration.

Example: The Hesse configuration

Let ε be a cubic root of unity in K.

$$\begin{array}{lll} (0,1,-1), & (1,0,-1), & (1,-1,0), \\ (0,1,-\varepsilon), & (1,0,-\varepsilon^2), & (1,-\varepsilon,0), \\ (0,1,-\varepsilon^2), & (1,0,-\varepsilon), & (1,-\varepsilon^2,0). \end{array}$$



3-nets and dual 3-nets

Definition: 3-nets (as abstract incidence structures)

A 3-net consists of a set $\mathcal{P}$ of points, three nonempty sets $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ of lines and an incidence relation $I \subset \mathcal{P} \times \mathcal{L}$ such that

- two lines from different classes are incident with a unique points, and,
- two lines from the same class are not incident with a common point.

Example: 3 line pencils.

Definition: Dual 3-nets (as abstract incidence structures)

A dual 3-net consists of three nonempty sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ of points, a set $\mathcal{L}$ of lines and an incidence relation $I \subset \mathcal{P} \times \mathcal{L}$ such that

- two points from different classes are connected by a unique line, and,
- two points from the same class are not connected by a line.

Terminology: The sets $\mathcal{P}_i$ are called fibers or components.

Algebraization of (dual) 3-nets

- For any (abstract) 3-net $|\mathcal{P}_1| = |\mathcal{P}_2| = |\mathcal{P}_3|$ holds.
- In case of a finite dual net, this number is the order.
- Let Q be a set with $|Q| = |\mathcal{P}_1| = |\mathcal{P}_2| = |\mathcal{P}_3|$ and let

  $$\alpha_i : Q \to \mathcal{P}_i$$

  be a bijection.
- For any $x, y \in Q$ there is a unique $z \in Q$ such that the points

  $$\alpha_1(x), \alpha_2(y), \alpha_3(z)$$

  are collinear.
- We define the binary operation $x * y = z$ on Q.
- Notice that 2 values of $\{x, y, z\}$ determine the third.

Quasigroups and projective realizations

Definition: Quasigroups

Let Q be a set with a binary operation $x * y$. $(Q, *)$ is a quasigroup if for any $a, b, c, d \in Q$, the equations

$$a * x = b, \qquad y * c = d$$

have unique solutions in x, y.

- Groups are precisely the associative quasigroups.

Definition: Projective realization of quasigroups

Let $(Q, *)$ be a quasigroup. We say that the maps

$$\alpha, \beta, \gamma : Q \to PG(2, K)$$

realize Q on the projective plane if the points $\alpha(x)$, $\beta(y)$, $\gamma(z)$ lie on a line if and only if $x * y = z$.

- The sets $\alpha(Q)$, $\beta(Q)$, $\gamma(Q)$ are fibers of an embedded dual 3-net.

Motivation and previous results

1. In this talk, we are interested in the projective realizations of finite groups.
2. Groups are treatable because the corresponding net has a rich subnet structure.
3. S. Yuzvinsky (Compos. Math. 2004) conjectured that only abelian groups can be realized.
4. Yuzvinsky also gave many existence and non-existence results over the base field $\mathbb{C}$.
5. J. Stipins (Arxiv, 2005) showed that the nonassociative quasigroup of order 5 can be realized.
6. G. Urzúa (Adv. Geom. 2010) classified the realizable quasigroups of order 6 and realized the quaternion group of order 8.
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Subnets and subgroups

- Let G be a group and let

$$\Lambda_1 = \alpha_1(G), \qquad \Lambda_2 = \alpha_2(G), \qquad \Lambda_3 = \alpha_3(G)$$

be a projective realization of G.

- Let H be a proper subgroup of G.
- Then, for any $a \in G \setminus H$,

$$\Delta_1 = \alpha_1(G), \qquad \Delta_2^a = \alpha_2(Ha), \qquad \Delta_3^a = \alpha_3(Ha)$$

is a projective realization of H.

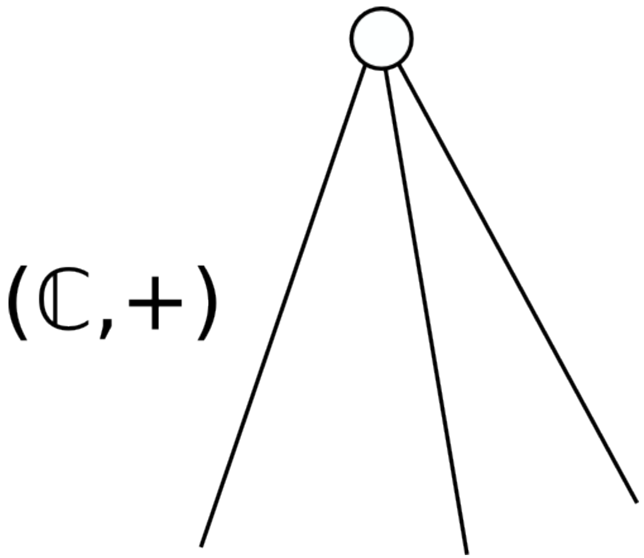
Description of the geometric structure by inductive argument

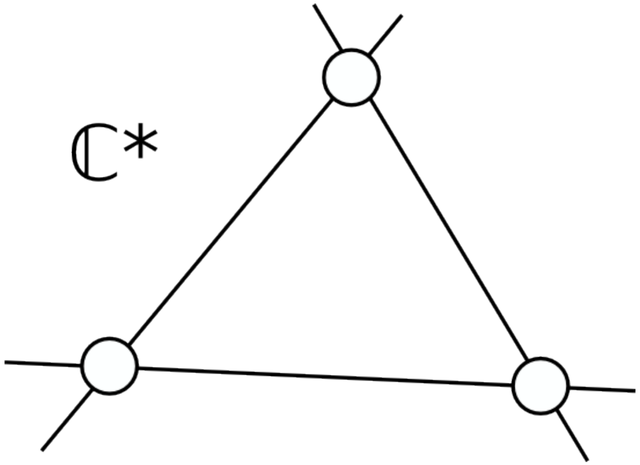
- Assume that the geometric structure of all realizations of H are known.
- H has several realizations sharing a fiber.
- Deduce global information. (???)

Dual 3-nets of "line type"

Definition: Dual 3-nets of "line type"

We say that a dual 3-net of $PG(2, \mathbb{C})$ is of line type if each fiber is contained in a line. If the lines have no point in common then the dual 3-net is called of triangular type.



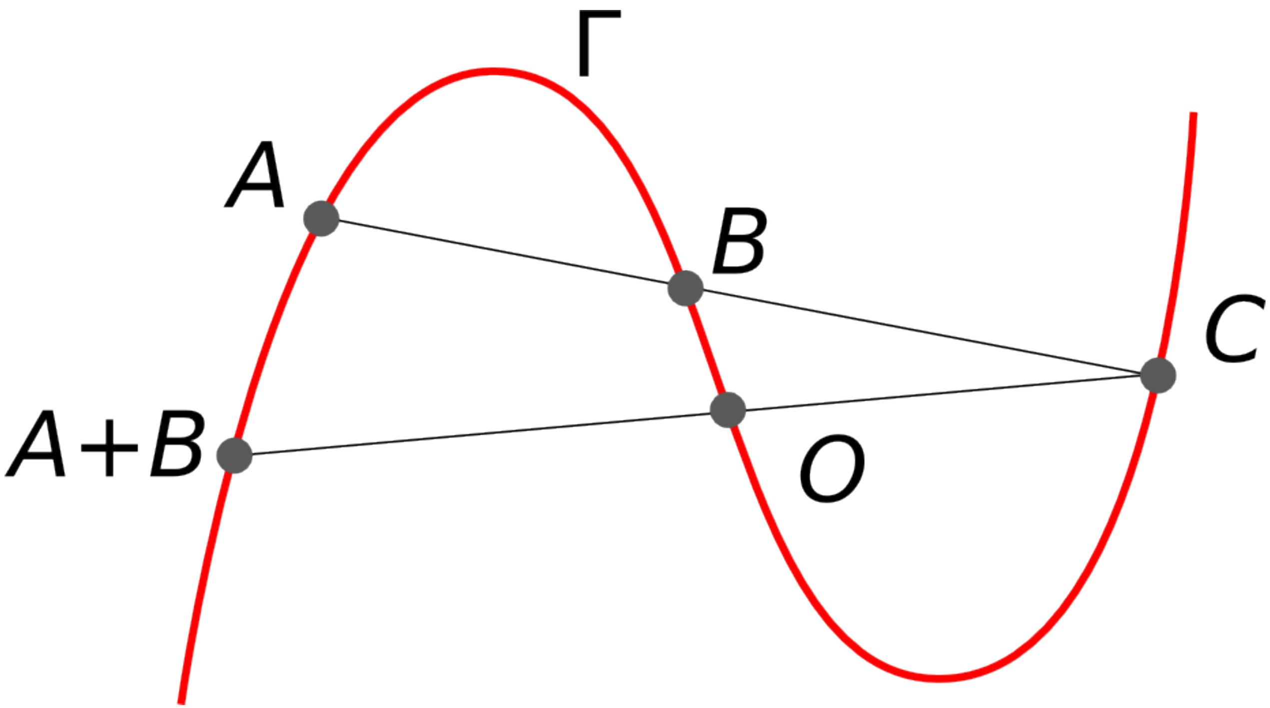


Remark. As $(\mathbb{C}, +)$ has no finite subgroups, the first type is not interesting for us.

The abelian group structure on the cubic curve

Theorem

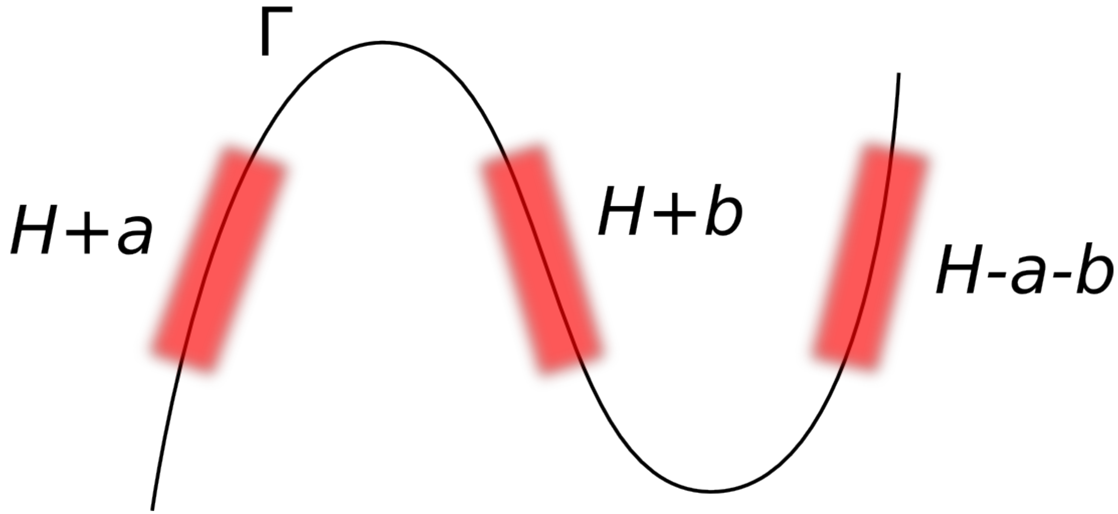
Let Γ be a nonsingular cubic curve. Then, we can define an abelian group $(\Gamma, +)$ in the following way.



Remark. If 0 is an inflexion point of Γ then the points $A, B, C \in \Gamma$ are collinear if and only if $A + B + C = 0$.

Dual 3-net realizations of "algebraic type"

Let Γ be a nonsingular cubic curve, O an inflexion point and H a (finite) subgroup of $(\Gamma, +)$. Then the cosets $H + a$, $H + b$, $H - a - b$ form a dual 3-net:



Definition: Algebraic dual 3-nets

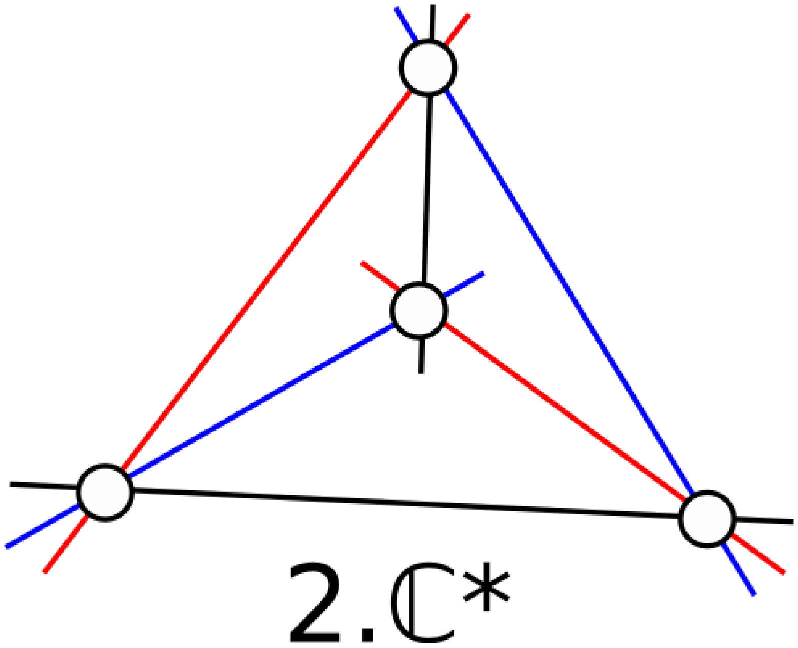
We say that a dual 3-net of $PG(2, \mathbb{C})$ is of algebraic type if all points are contained in a cubic curve.

Remark. Line type is also algebraic.

Dual 3-nets of “tetrahedron type”

Definition: Dual 3-nets of “line type”

We say that a dual 3-net of $PG(2,\mathbb{C})$ is of tetrahedron type if it is contained in the following configuration of six lines.



Proposition (KNP 2011)

Tetrahedron type dual 3-nets correspond to dihedral groups.

The main result

Main Theorem (Korchmáros, Nagy, Pace 2012)

Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order $n \geq 4$ in the projective plane $PG(2, \mathbb{C})$ which realizes a group G. Then one of the following holds.

(I) G is either cyclic or the direct product of two cyclic groups, and $(\Lambda_1, \Lambda_2, \Lambda_3)$ is algebraic.

(II) G is dihedral and $(\Lambda_1, \Lambda_2, \Lambda_3)$ is of tetrahedron type.

(III) G is the quaternion group of order 8.

(IV) G has order 12 and is isomorphic to Alt_4.

(V) G has order 24 and is isomorphic to Sym_4.

(VI) G has order 60 and is isomorphic to Alt_5.

Remark. Computer calculations show that Alt_4 has no projective realization. This implies that the cases (IV)-(VI) cannot actually occur.

Step 1: The cyclic case

Proposition (Yuzvinsky, KNP)

Any dual 3-net realizing a cyclic group is of algebraic type.

The proof uses the theorem of Lamé from algebraic geometry.

Proposition (Yuzvinsky)

- If an abelian group G contains an element of order ≥ 10 then every dual 3-net realizing G is algebraic.
- No dual 3-net realizes an elementary abelian group of order 2^h with $h \geq 3$.

Proposition (Blokhuis, Korchmáros, Mazzocca)

If the fiber Λ_1 is contained in a line then $\Lambda_2 \cup \Lambda_3$ is contained in a conic.

Step 2: The cyclic normal subgroup case

Proposition

- Let G be a finite group containing a normal subgroup H of order $n \geq 3$.
- Assume that G can be realized by a dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ and that every dual 3-subnet of $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizing H as a subgroup of G is triangular.
- Then H is cyclic and $(\Lambda_1, \Lambda_2, \Lambda_3)$ is either triangular or of tetrahedron type.

Step 3: Central homologies preserving the fibers

Proposition

- Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order $n \geq 4$ realizing a group G.
- If every point in Λ_1 is the center of an involutory homology which preserves Λ_1 while interchanges Λ_2 with Λ_3,
- then either Λ_1 is contained in a line, or $n = 9$.
- In the latter case, $(\Lambda_1, \Lambda_2, \Lambda_3)$ lies on a non-singular cubic Γ whose inflection points are the points in Λ_1.

Step 4: Central homologies preserving the fibers

Proposition

Let G be a group containing a proper abelian subgroup H of order $n \geq 5$. Assume that a dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizes G such that all its dual 3-subnets $(\Gamma_1^j, \Gamma_2, \Gamma_3^j)$ realizing H as a subgroup of G are algebraic.

Let Γ_j be the cubic through the points of $(\Gamma_1^j, \Gamma_2, \Gamma_3^j)$. If $(\Lambda_1, \Lambda_2, \Lambda_3)$ is not algebraic then Λ_2 contains three collinear points and one of the following holds:

(i) Λ_2 is contained in a line.

(ii) $n = 5$ and there is an involutory homology with center in Λ_2 which preserves every Γ_j and interchanges Λ_1 and Λ_3.

(iii) $n = 6$ and there are three involutory homologies with center in Λ_2 which preserves every Γ_j and interchanges Λ_1 and Λ_3.

(iv) $n = 9$ and Λ_2 consists of the nine common inflection points of Γ_j.

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Dual k-nets in projective planes

Proposition (KNP, 2013)

Every dual 3-net has a constant cross-ratio κ. Moreover, $\kappa^{n(n-1)} = (\kappa-1)^{n(n-1)} = 1$ holds.

Theorem (Stipins, Yuzvinsky, KNP)

If $p = 0$ or $p > 3^{\varphi(n(n-1))}$ then $\kappa^2 - \kappa + 1 = 0$. In particular, in this case no dual k-nets exist for $k > 4$.

Further results:

- Description of the geometry of k-nets ($k \geq 4$) with a fiber contained in a line.
- Example of dual $(q+1)$-net in $PG(2, q^s)$, $s \geq 3$. [Idea due to Lunardon.]

Open questions

1. Projective realizations over (algebraically closed) fields of small characteristic.
2. Projective realization of infinite classes of non-associative quasigroups.
3. Dual 4-nets in projective planes.
4. The geometric description of the realization of Q_8.

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THANK YOU FOR YOUR ATTENTION!