Regular polygons revisited

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June 13, 2013

Finite Geometry Conference and Workshop
University of Szeged
10—14 June, 2013

Abstract

Basic facts about regular polygons, and the notion of regularity, are well known since the beginning of 70's of last century. Starting with the theorem about a spatial regular pentagon being planar (Van der Waerden, 1970), a whole theory has been built up, mainly in the n-dimensional Euclidean space. Total regularity implies a nice behaviour of the k-gon, depending on the parity of k. Via different models and techniques, similar theorems on properties and classifications were discovered, then rediscovered independently. The very elementary geometric question wether a regular $(n+1)$-gon spans the n-dimensional space, and under what conditions, drew the attention of geometers again and again during last four decades. The same theorems were discovered several times independently, in different interpretations.

Abstract

In an early article, Gabor Korchmaros used geometric transformations to solve the problem completely in three-dimensional spaces. The method is of absolute character, so the result is valid not only in Eucliden space but in absolute geometry, as well. Our efforts for generalizing these results for higher dimensional spaces, lead to some results, already known, however the transformation technics would help us to understand and retrieve the deeper geometric relations.

Grünbaum

Branko Grünbaum published a general survey on polygons, dedicated to Leonard M. Blumenthal, in 1975. (Polygons. "The Geometry of Metric and Linear Spaces", L. M. Kelly, ed. Lecture Notes in Mathematics Number 490, pp. 147 - 184. Springer-Verlag, Berlin-Heidelberg-New York 1975.)
Its third section deals with 'Equilateral polygons', not restricted to be planar. His remark on the kind of regularity, is very honest and gives a realistic evaluation of different efforts:

> *Actually, we are concerned with a number of distinct topics, depending on the kind of regularity we wish to consider. Most of the notions we shall discuss were discovered several times, the authors usually not being aware of the relevabt work of the others.*

Auric (1911)

Definition

An n-gon P is called *regular* provided it is *equilateral* and *isogonal*. I.e. all edges of P have the same length and the angles between adjacent edges are all equal.

Statement

Each regular pentagon is planar.

Valiron (1911)

Statement

Skew regular n-gons exist for each odd $n > 5$.

Remark

Existence of skew regular n-gons for even $n \geq 4$ is obvious.

conjecture

If the oriented dihedral angles at each edge are equal then the regular n-gon is planar (E^3).

Generalisation

The generalisation of the conjecture above for all dimension and a proof was given by Coxeter (1974), independently.

Regular polygons of crown type

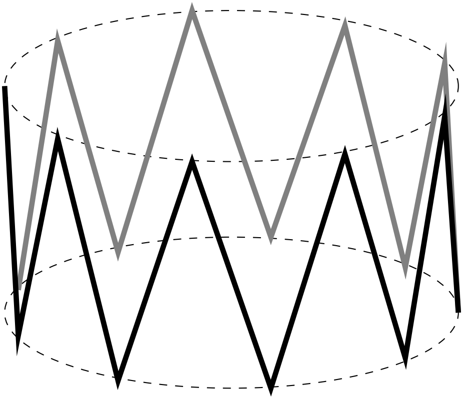


Figure: Regular polygon of crown type, n is even

Arnold (1957)

Question

For which n does there exist a spatial n-gon with all its sides of the same length a and all angles of the same magnitude α?

Answer

Partly answered by A.P. Garber — V.I. Garvackij — V.J. Jarmolenko (1962).

Answer; Grünbaum (1975), Proposition 8.

For each α with $0 < \alpha < \alpha_k = \frac{(k-2)\pi}{k}$ there exist equilateral skew n-gons isogonal with angle α for each even $n \geq k \geq 4$.
For each $0 < \alpha < \alpha_k$ there exist equilateral skew n-gons isogonal with angle α for each odd $n \geq \max\{k, 7\}$.
There exist no skew equilateral and isogonal pentagons.

Regular pentagons

- Independent reapparence in 1970.
- Problem arose by interest of organic chemists.

Theorem. (van der Waerden [1970])

A spatial pentagon $ABCDE$ in which all sides equal a and all angles equal α is planar.

Several other proofs

- Lüssi — Trost (1970)
- Irminger (1970)
- Dunitz — Waser (1972)
- van der Waerden (1972)
- Smakal (1972)
- Kárteszi (1973)
- Bottema (1973)

O. Bottema (1973) on regular pentagons

Theorem

The necessary and sufficient condition for the existence of a skew equilateral pentagon that spans E4 and is isogonal with angle α is

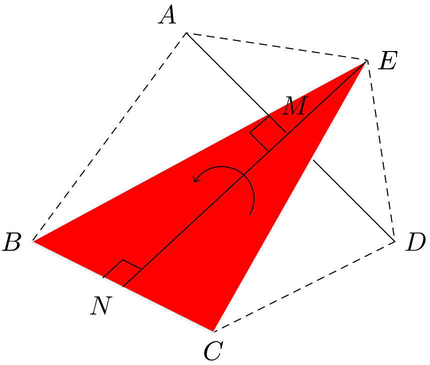
$$\frac{pi}{5} < \alpha < \frac{3\pi}{5}.$$

- For $\alpha = \frac{pi}{5}$ we have the regular pentagram (planar),
- for $\alpha = \frac{3\pi}{5}$ we have the regular pentagon (planar).

The point of the proof of van der Waerden

Lemma

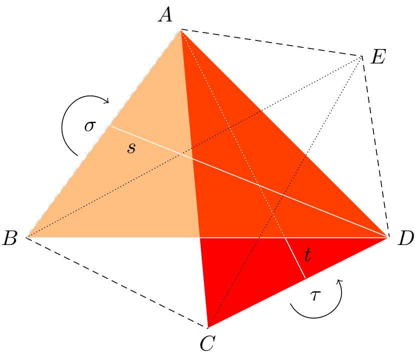
If four points $ABCD$ of the 5 points of the regular pentagon $ABCDE$ are not in a plane then there exists a reflection τ in an axis (halfturn)leaving E fixed and interchanging A with D, and B with C.



The point of the proof of van der Waerden [2]

There are 3 cases

1. All 5 points are in a plane.
2. 4 out of the 5 vertices (say $ABCD$) are in a plane, however, E is not in that plane.
3. No 4 out of the 5 vertices are in one plane.



$$
\begin{array}{rccccc}
 & A & B & C & D & E \\
\tau: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & A & E & D & D & B \\
\sigma: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & B & C & D & E & A
\end{array}
$$

Completing the proof

- So that $\varrho = \sigma \circ \tau$ generates a cyclic permutation

$$\begin{pmatrix} A & B & C & D & E \\ B & C & D & E & A \end{pmatrix}.$$

- The symmetry ϱ fixes the center of mass S of the pentagon.
- Product of halfturns around t and s has a fixed point.
- So ϱ is neither a skrew motion nor a translation.
- Axes s and t have a point of intersection.
- ϱ is a rotation around an axis m perpendicular to both s and t at their point of intersection S.

Notion of regularity of n-gons

A set of (differrent) points $\{P_1, P_2, \dots, P_n\}$ we shall call a *regular n-gon* if

$$d(P_1, P_{1+k}) = d(P_i, P_{i+k}) \quad \text{for} \quad 1 \le k \le n-1, \quad i = 1, 2, \dots, n,$$

where the indices are taken mod n.

Notation

Points of E^r will be considered vectors of components r, and denoted as follows. A point P will be identified with vector

$$\mathbf{p} = (p_1, \dots, p_r).$$

On special choice of the the origin

Since the regularity is defined by distances of vertices, and a translation dos not changes these distances, we can translate our

Regular polygons in r-dimensional Euclidean spaces [2]

Lemma on extensions of symmetries of a regular polygon

Regularity conditions provide us with specific symmetries of our regular polygon. Furthermore, these symmetries can be extended to symmetries of the space $\mathbf{E^r}$ itself.

Lemma

There is an isometric mapping φ of $\mathbf{E^r}$ onto itself, such that for the set of the vertices $\mathbf{V} = [\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}]$ of a regular polygon $\varphi(\mathbf{v_i}) = \mathbf{v}_{\mathrm{i}+1} \mod n$.

Proof

Regular polygon of vertices $[\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}]$, by definition, fulfill all conditions

$$d(\mathbf{v_i}, \mathbf{v_j}) = d(\mathbf{v_{i+m}}, \mathbf{v_{j+m}})$$

for all $1 \leq i < j \leq n$, where $i + m = i' \mod n$. So for $m = 1$ we have a mapping

cont. proof

$$f : \mathbf{V} \to \mathbf{V}; \mathbf{v_1} \mapsto \mathbf{v_2}, \mathbf{v_2} \mapsto \mathbf{v_3}, \ldots, \mathbf{v_n} \mapsto \mathbf{v_1}.$$

such that each pair of vertices is mapped onto a pair of vertices of the same distance. In the sense of Definition 12.2. of [1] mapping f is a *congruent (or isometric) mapping*[a] *of* $\mathbf{V}$ *onto* $\mathbf{V}$.

However, Property IV. of [1] says that *Any congruence between any two subsets of* E_n *[the n-dimensional Euclidean space] can be extended to a motion*[b], so it applies to our mapping f, and we can extend it to an isometry φ of $\mathbf{E^r}$.

[a] L.M. Blumenthal, Theory and Applications of Distance Geometry, 35 p.
[b] L.M. Blumenthal, Theory and Applications of Distance Geometry, 93 p.

Now we have a special isometry φ of $\mathbf{E^r}$ which is a symmetry of $\mathbf{V}$: φ performs a cyclic move of the polygon, sending each vertex to the next one (in the cyclic order of the vertices).

Lemma

If $\mathbf{V} = [\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}]$ is a set of vertices of a regular polygon, and φ is an isometry of $\mathbf{E^r}$ such that φ is a symmetry of $\mathbf{V}$, then with appropriate chose of the origin $\mathbf{0}$ of $\mathbf{E^r}$, φ fixes it.

Representation of φ in matrix form

φ fixes the origin, so this mapping is a linear isometry of $\mathbf{E^r}$. It is well known that by a proper choice of the basis for $\mathbf{E^r}$, the $r \times r$ matrix $\mathbf{F}$ of φ has the canonical form

$$F = \begin{pmatrix} \pm 1 & & & & & & \\ & 1 & & & & \Large 0 & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & \Theta_1 & & \\ & \Large 0 & & & & \ddots & \\ & & & & & & \Theta_k \end{pmatrix}, \qquad (1)$$

where the first entry is -1 if the isometry is indirect, furthermore Θ_i (for $i = 1, \ldots, k$) are the matrices of rotations of a 2-dimensional subspace through angle ϑ_i:

$$\Theta_i = \begin{pmatrix} \cos\vartheta_i & -\sin\vartheta_i \\ \sin\vartheta_i & \cos\vartheta_i \end{pmatrix}.$$

So that $r = 2k + m$ where k, m are integers with $0 \leq k \leq r/2$, $0 \leq m \leq r$, and the number of 1-s is m if φ is direct (the first entry is $+1$) and the number of 1-s is $m - 1$ if φ is an indirect isometry (the first entry is -1).
This matrix representation shows that $\mathbf{E^r}$ is direct sum of pairwise orthogonal subspaces, each one is of dimension 1 or 2 and fixed by φ.

Some basic properties of **V** can be obtained by analysing its symmetries.
Case 1 $m \geq 1$.

$$F^2 = \begin{pmatrix} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & \ddots & & & & & 0 & & \\ & & & 1 & & & & & & \\ & & & & \cos 2\vartheta_1 & -\sin 2\vartheta_1 & & & & \\ & & & & \sin 2\vartheta_1 & \cos 2\vartheta_1 & & & & \\ & & & & & & \ddots & & & \\ & & 0 & & & & & \cos 2\vartheta_k & -\sin 2\vartheta_k \\ & & & & & & & \sin 2\vartheta_k & \cos 2\vartheta_k \end{pmatrix}.$$

In this case at least one of these φ-invariant subspaces has dimension 1, i.e. number m of (± 1)-s is greater or equal to 1, then φ^2 has the form
The 1-dimensional subspace $\mathbf{Z_1}$ generated by vector $\mathbf{e_1} = (1, 0, \ldots, 0)$, is fixed pointwise at φ^2 since it has two fixed points: $\mathbf{e_1}$ and $\mathbf{0}$.
As for any vertex $\mathbf{v_i}$ $(1 \leq i \leq n)$ of the polygon

$$\mathbf{V_i} = \{(\varphi^2)^s(\mathbf{v_i}) | s \in \mathbf{N}\} =$$

$$= \begin{cases} \{\mathbf{v_{i+2}}, \mathbf{v_{i+4}}, \ldots, \mathbf{v_{i-1}}, \mathbf{v_{i+1}}, \mathbf{v_{i+3}}, \mathbf{v_{i-2}}, \mathbf{v_i}\} = \mathbf{V} & \text{if } n \text{ is odd} \\ \{\mathbf{v_{i+2}}, \mathbf{v_{i+4}}, \ldots, \mathbf{v_{i-2}}, \mathbf{v_i}\}, \text{i.e. every second vertex} & \text{if } n \text{ is even,} \end{cases}$$

in case of $\mathbf{v_i} \in \mathbf{Z_1}$ for some i, all vertices, or half of them are the same point, belonging to $\mathbf{Z_1}$. We excluded this degenerated case at the beginning, so we we can suppose that the polygon has no vertex in $\mathbf{Z_1}$.

Now we observe how the set $\mathbf{V_i}$ of vertices is situated in space $\mathbf{E^r}$.

Lemma

Set $\mathbf{V}_i$ of vertices above, is in an affine subspace of dimension less than r, orthogonal to the 1-dimensional subspace $\mathbf{Z_1}$.

Proof

It is enough to prove the statement for set $\mathbf{V_1}$, generated by vertex $\mathbf{v_1}$, since $\mathbf{V_i}$ is generated by each element from
Any vector $\mathbf{v_i}$ can be expressed in the form

$$\mathbf{v_i} = \mathbf{u_i} + \mathbf{u_i}^{\perp}, \quad \mathbf{u_i} \in \mathbf{Z_1}, \quad \mathbf{u_i}^{\perp} \in \mathbf{Z_1}^{\perp},$$

where $\mathbf{Z_1}^{\perp}$ is the orthogonal complementer subspace of $\mathbf{Z_1}$.

Let we start from the equation

$$\mathbf{v_1} - \mathbf{u_1} = \mathbf{u_1}^{\perp}.$$

The inner product with $\mathbf{u_1}$ gives

$$\mathbf{u_1}(\mathbf{v_1} - \mathbf{u_1}) = \mathbf{u_1}\mathbf{u_1}^{\perp} = 0,$$

consequently

$$\mathbf{u_1}\mathbf{v_1} = (\mathbf{u_1})^2.$$

However, by the orthogonality of φ, the same is true for $(\varphi^2)^j$ (for any j), so

$$(\varphi^2)^j(\mathbf{u_1})(\varphi^2)^j(\mathbf{v_1}) = \left((\varphi^2)^j(\mathbf{u_1})\right)^2.$$

Furthermore, $(\varphi^2)^j$ fixes $\mathbf{u_1}$ as a vector in $\mathbf{Z_1}$, while sends $\mathbf{v_1}$ into $\mathbf{v_{1+2j}}$:

$$\mathbf{u_1}\mathbf{v_{1+2j}} = (\mathbf{u_1})^2 \quad \text{for all } j.$$

We arrived at the conclusion that all vertices $\mathbf{v_{1+2j}}$ have the same orthogonal component along $\mathbf{Z_1}$, i.e. all these vertices are in the same affine subspace $\mathbf{v_1} + \mathbf{Z_1}^{\perp}$, orthogonal to $\mathbf{Z_1}$.

Conclusions [1]

The lemma above says that in this case the polygon is

- either in a lower dimension affine subspace of $\mathbf{E^r}$ (when n is odd),
- or the vertices with index of the same parity are in a lower dimension affine subspace of $\mathbf{E^r}$ (when n is even), orthogonal to the same 1-dimensional subspace.

In the case of odd n, the baricenter of the polygon is in the same $(r-1)$-dimensional affine subspace as all vertices of the polygon, so $\mathbf{0}$ is in that affine subspace, and the polygon is in an $(r-1)$-dimensional linear subspace as well.
If the dimension r of $\mathbf{E^r}$ is odd, then m is an odd number as well, so all the above results apply.

Conclusions [2]

Theorem

In an odd dimensional space $\mathbf{E^r}$ *any regular polygon with vertices of odd number, is in a lower dimensional affine subspace. If the number of vertices is even then the vertices with even (resp. odd) indices are contained in a proper subspace of* $\mathbf{E^r}$. *Then, either all vertices lie in the same proper subspace or the polygon is of crown type.*

Conclusions [3]

Repeating for the 1-dimensional subspaces $\mathbf{Z_2}, \ldots, \mathbf{Z_m}$, obtain that $\mathbf{V_i}$ is in the intersection of m affine subspaces, each is of dimension $(r-1)$. We can add this to the previous theorem.

Theorem

If φ has m different 1-dimensional invariant subspaces, then the polygon is in an $(r-m)$-dimensional linear subspace, orthogonal to an m-dimensional subspace of $\mathbf{E^r}$ when n is odd, or the vertices with indices of the same parity are in an $(r-m)$-dimensional affine subspace, orthogonal to an m-dimensional subspace of $\mathbf{E^r}$ when n is even.

Case 2: $m = 0$. In this case dimension r of $\mathbf{E^r}$ is even.
Let vertex P of our regular n-gon has coordinate vector $\mathbf{p} = (p_1, p_2, \ldots, p_r)$, with respect to our coordinate system. So for $\mathbf{p}$ and the matrix form $\mathbf{F}$ of the isometry φ the following equation holds: $\mathbf{F^t p^t} =$

$$\begin{pmatrix} \cos\vartheta_1 & \sin\vartheta_1 & & & & \\ -\sin\vartheta_1 & \cos\vartheta_1 & & & \text{0} & \\ & & \cos\vartheta_{2i} & \sin\vartheta_{2i} & & \\ & & -\sin\vartheta_{2i} & \cos\vartheta_{2i} & & \\ \text{0} & & & & \cos\vartheta_{r/2} & \sin\vartheta_{r/2} \\ & & & & -\sin\vartheta_{r/2} & \cos\vartheta_{r/2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{2i-1} \\ p_{2i} \\ \vdots \\ p_{r-1} \\ p_r \end{pmatrix}.$$

Classification via structure of $\mathbf{F}$; Case 2; cont.

Then the consecutive vertices $P, \varphi(P), \varphi^2(P), \ldots, \varphi^{r-1}(P)$ are contained in a proper subspace of $\mathbf{E^r}$ if and only if the vectors

$$\mathbf{p^t}, \mathbf{F^t p^t}, (\mathbf{F^t})^{\mathbf{2}}\mathbf{p^t}, \ldots, (\mathbf{F^t})^{\mathbf{r-1}}\mathbf{p^t}$$

are linearly dependent. Such a linear dependence comes true whenever $p_{2i-1} = p_{2i} = 0$ for at least one i with $0 \le i \le r/2 - 1$, since each vector, according to the previous matrix product, has the form $(p_1', p_2', \ldots, p_{2i_2}', p_{2i-1}, 0, 0, \ldots, p_{r-1}, p_r)$. I.e. all these vectors are in an $(r-2)$-dimensional subspace. From now on we exclude this particular case.

Non-generated regular polygons [1]

Now we are ready to derive a necessary and sufficient condition for a regular polygon being non-degenerated in an even dimensional Euclidean space.
Let us group the coordinates of point P by two to get a complex representation of it in the following form

$$\tau_1 = p_1 + ip_2, \quad \tau_2 = p_3 + ip_4, \quad \ldots, \quad \tau_{r/2} = p_{r-1} + ip_r.$$

This is given by a bijective map

$$c : \mathbf{E^r}(\cong \mathbf{R^r}) \to \mathbf{C^{r/2}} : (p_1, p_2, \ldots, p_{r-1}, p_r) \mapsto (p_1+ip_2, \ldots, p_{r-1}+ip_r),$$

which is an isomorphism between the two vector spaces.

Then consider the $(r/2)\times(r/2)$ complex diagonal matrix

$$\mathbf{F}_{\mathbf{C}} = \begin{pmatrix} \cos\vartheta_1 + i\sin\vartheta_1 & & 0 \\ & \ddots & \\ 0 & & \cos\vartheta_{r/2} + i\sin\vartheta_{r/2} \end{pmatrix}$$

It is clear that $\quad \mathbf{F}_{\mathbf{C}} = \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_{r/2} \end{pmatrix}$

$$= \begin{pmatrix} (\cos\vartheta_1 p_1 - \sin\vartheta_1 p_2) + i(\sin\vartheta_1 p_1 + \cos\vartheta_1 p_2) \\ \vdots \\ (\cos\vartheta_{r/2} p_{r-1} - \sin\vartheta_{r/2} p_r) + i(\sin\vartheta_{r/2} p_{r-1} + \cos\vartheta_{r/2} p_r) \end{pmatrix},$$

Non-generated regular polygons [3]

$$\mathbf{F}_{\mathbf{C}}^{\mathbf{k}} \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_{r/2} \end{pmatrix} =$$

$$\begin{pmatrix} \cos k\vartheta_1 + i \sin k\vartheta_1 & & 0 \\ & \ddots & \\ 0 & & \cos k\vartheta_{r/2} + i \sin k\vartheta_{r/2} \end{pmatrix} \begin{pmatrix} p_1 + ip_2 \\ \vdots \\ p_{r-1} + ip_r \end{pmatrix}$$

$$= \begin{pmatrix} (\cos k\vartheta_1 p_1 - \sin k\vartheta_1 p_2) + i(\sin k\vartheta_1 p_1 + \cos k\vartheta_1 p_2) \\ \vdots \\ (\cos k\vartheta_{r/2} p_{r-1} - \sin k\vartheta_{r/2} p_r) + i(\sin k\vartheta_{r/2} p_{r-1} + \cos k\vartheta_1 p_r) \end{pmatrix},$$

for $1 \leq k \leq r/2$.

and that

$$\mathbf{F}_{\mathbf{C}}^{\mathbf{k}} \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_{r/2} \end{pmatrix} =$$

$$\begin{pmatrix} \cos k\vartheta_1 + i \sin k\vartheta_1 & & 0 \\ & \ddots & \\ 0 & & \cos k\vartheta_{r/2} + i \sin k\vartheta_{r/2} \end{pmatrix} \begin{pmatrix} p_1 + ip_2 \\ \vdots \\ p_{r-1} + ip_r \end{pmatrix} =$$

$$\begin{pmatrix} (\cos k\vartheta_1 p_1 - \sin k\vartheta_1 p_2) + i(\sin k\vartheta_1 p_1 + \cos k\vartheta_1 p_2) \\ \vdots \\ (\cos k\vartheta_{r/2} p_{r-1} - \sin k\vartheta_{r/2} p_r) + i(\sin k\vartheta_{r/2} p_{r-1} + \cos k\vartheta_1 p_r) \end{pmatrix},$$

for $1 \leq k \leq r/2$.

Non-generated regular polygons [5]

If we introduce the notation $\zeta_j^k = \cos k\vartheta_j + i \sin k\vartheta_j$ with $j = 1, \ldots, r/2$, then the previous equation takes the form

$$\mathbf{F}_{\mathbf{C}}^{\mathbf{k}} \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_{r/2} \end{pmatrix} = \begin{pmatrix} \zeta_1^k \tau_1 \\ \vdots \\ \zeta_{r/2}^k \tau_{r/2} \end{pmatrix} \qquad (0 \leq k \leq r/2 - 1).$$

Rephrasing linear dependence

Now, the linear dependence of the vectors, corresponding the first $(r/2)$ consecutive vertices of the polygon and the linear dependence of the $r/2$ vectors on the right hand side of the above equation is equivalent, and can be rephrased as the linear dependence of the row vectors of the matrix

$$G = \begin{pmatrix} \tau_1 & \tau_2 & \cdots & \tau_{r/2} \\ \zeta_1\tau_1 & \zeta_2\tau_2 & \cdots & \zeta_{r/2}\tau_{r/2} \\ \zeta_1^2\tau_1 & \zeta_2^2\tau_2 & \cdots & \zeta_{r/2}^2\tau_{r/2} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1^{r/2}\tau_1 & \zeta_2^{r/2}\tau_2 & \cdots & \zeta_{r/2}^{r/2}\tau_{r/2} \end{pmatrix}.$$

Criteria

Since we have excluded the case $p_{2i-1} = p_{2i} = 0$ $(i = 1, \ldots, r/2)$, we have $\tau_i \neq \mathbf{0}$, so

$$\det(G) = \tau_1 \tau_2 \ldots \tau_{r/2} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \zeta_1 & \zeta_2 & \ldots & \zeta_{r/2} \\ \zeta_1^2 & \zeta_2^2 & \ldots & \zeta_{r/2}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1^{r/2} & \zeta_2^{r/2} & \ldots & \zeta_{r/2}^{r/2} \end{vmatrix}.$$

From the classical result on Vandermonde determinant, we have:

Conclusion

G has maximum rank if and only if the complex numbers ζ_j are pairwise distinct. I.e. the angles ϑ_j are pairwise distinct $\mod 2\pi$.

The n-gon $\mathcal{P} = P_0 P_1 \cdots P_{n-1}$ is either degenerate or of crown type for r odd, while it is non-degenerate, apart a few exceptions, for r even.