Semiovals and semiarcs

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The beginning, semi-quadratic sets

Semiovals first appeared as special examples of *semi-quadratic sets*. Let Π be a projective space and $\mathcal{Q} = (\mathcal{P}, \mathcal{L})$ be a pair consisting of a set $\mathcal{P}$ of points of Π, and a set $\mathcal{L}$ of lines of Π. A *tangent* to $\mathcal{Q}$ at $P \in \mathcal{P}$ is a line $\ell \in \mathcal{L}$ such that P is on ℓ, and either $\ell \cap \mathcal{P} = \{P\}$, or $\ell \in \mathcal{L}$. $\mathcal{Q}$ is called semi quadratic set (SQS), if every point on a line of $\mathcal{L}$ belongs to $\mathcal{P}$, and for all $P \in \mathcal{P}$ the union $\mathcal{T}_P$ of all tangents to $\mathcal{Q}$ at P is either a hyperplane or the whole space Π. A lot of attempts were made to classify all SQS, but the problem is still open in general.

An SQS $\mathcal{Q} = (\mathcal{P}, \mathcal{L})$ is called a *semi-ovoid* (or *semioval* if dim $\Pi = 2$), if $\mathcal{L} = \emptyset$ and $\mathcal{P}$ contains at least 2 points. The complete characterization of semi-ovoids was given by J. Thas. Using elementary double counting arguments, he proved the following results.

Semi-ovoids

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Theorem

- *The only semi-ovoids of* $PG(3, q)$ *are the ovoids (set of* $q^2 + 1$ *points, no three of them are collinear).*
- *In* $PG(n, q)$, $n > 3$, *there are no semi-ovoids.*

Semiovals

In the planar case the situation is much more complicated. It is easy to see, that the following simpler definition of semiovals is equivalent to the previously given one.

Definition

Let Π_q be a projective plane of order q. A *semioval* in Π_q is a non-empty pointset $\mathcal{S}_1$ with the property that for every point $P \in \mathcal{S}_1$ there exists a unique line t_P such that $S \cap t_P = \{P\}$. This line is called the tangent to $\mathcal{S}_1$ at P.

Definition

Let Π_q be a projective plane of order q. A non-empty pointset $\mathcal{S}_t \subset \Pi_q$ is called a t-semiarc if for every point $P \in \mathcal{S}_t$ there exist exatly t lines $\ell_1, \ell_2, \ldots \ell_t$ such that $\mathcal{S}_t \cap \ell_i = \{P\}$ for $i = 1, 2, \ldots, t$. These lines are called the tangents to $\mathcal{S}_t$ at P.

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If a line ℓ meets $\mathcal{S}_t$ in $2, 3$ or $3 < k$ points, then ℓ is called bisecant, trisecant or k-secant of $\mathcal{S}_t$, respectively.

Examples

Example

- k-arcs, $t = q + 2 - k$.
- Semiovals, $t = 1$.
- Subplanes, $t = q - m$, where m is the order of the subplane.

More examples

Proposition

Let $\mathcal{S}_t$ be a t-semiarc in Π_q. The followings hold:

- *if $t = q + 1$, then $\mathcal{S}_t$ is a single point,*
- *if $t = q$, then $\mathcal{S}_t$ is a subset of a line, and vice versa any subset of a line containing at least two points is a q-semiarc,*
- *if $t = q - 1$, then $\mathcal{S}_t$ is a set of three non-collinear points.*

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There exist t-semiarcs for each value of t satisfying $1 \leq t < q - 1$.

Example

Let ℓ_1 and ℓ_2 be two lines of Π_q, and let $1 \leq t < q - 1$ be an arbitrary integer. If we delete the point $\ell_1 \cap \ell_2$ and t other points from both lines, then the remaining $2(q - t)$ points obviously form a t-semiarc.

Classification?

Hopeless. We have to add extra conditions.

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Inequalities

Proposition

If $\mathcal{S}_t$ is a t-semiarc in Π_q, then $q - t + 2 \leq |\mathcal{S}_t|$.

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Equality holds for any $(q - t + 2)$-arc in Π_q.

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Theorem (Inequality 8)

If $\mathcal{S}_t$ is a t-semiarc in Π_q, then

$$|\mathcal{S}_t| \leq 1 + \left\lfloor \frac{q(t - 1 + \sqrt{4tq - 3t^2 + 2t + 1})}{2t} \right\rfloor .$$

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Equality holds for some values:
If $\mathcal{S}_t$ is a unital, then $t = 1$ and $|\mathcal{S}_t| = q\sqrt{q} + 1$;
if $\mathcal{S}_t$ is a Baer-subplane, then $t = q - \sqrt{q}$ and $|\mathcal{S}_t| = q + \sqrt{q} + 1$.

Regular semiovals

The notion of regular semioval was introduced by de Finis.

Definition

Let $\mathcal{S}_1$ be a semioval in Π_q. If all nontangent lines intersect $\mathcal{S}_1$ in either 0 or a constant number a of points, then $\mathcal{S}_1$ is called *regular semioval with character a*.

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Blokhuis and Szőnyi proved the following results.

Theorem

Let $\mathcal{S}_1$ be a regular semioval with character a in Π_q. Then $\mathcal{S}_1$ is an oval (thus $a = 2$), or a divides $q - 1$ and the points not on $\mathcal{S}_1$ are on 0 or on a tangents.

A consequence of this theorem is that the tangents of $\mathcal{S}_1$ form a regular semioval on the dual plane of Π_q. (This was proved by de Finis, too.)

Regular semiovals

Theorem (A. Blokhuis, T. Szőnyi, 1992)

Let $\mathcal{S}_1$ be a regular semioval with character a in $PG(2,q)$. Then there are two possibilities:

- *$\mathcal{S}_1$ is a unital (thus $a = \sqrt{q} + 1$);*
- *$a - 1$ and q are coprimes, and the tangents at collinear points of $\mathcal{S}_1$ are concurrent.*

The longstanding regular semioval conjecture in the Desarguesian planes was finally proved by Gács.

Theorem (A. Gács, 2006)

Let $\mathcal{S}_1$ be a regular semioval in $PG(2,q)$. Then $\mathcal{S}_1$ is either an oval or a unital.

Regular semiarcs

How can we define?

Definition

Let $\mathcal{S}_t$ be a t-semiarc in Π_q. If all nontangent lines intersect $\mathcal{S}_t$ in either 0 or a constant number a of points, then $\mathcal{S}_t$ is called *regular semiarc with character* a.

Regular semiarcs

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Let $\mathcal{S}_t$ be a t-semiarc in Π_q. If all nontangent lines intersect $\mathcal{S}_t$ in either 0 or a constant number a of points, then $\mathcal{S}_t$ is called *regular semiarc with character* a.

Examples: subplanes

Regular semiarcs

Another possible definition:
$\mathcal{S}_t$ is regular if equality holds in Inequality (8), hence $|\mathcal{S}_t|$ is maximal.

Regular semiarcs

Another possible definition:
$\mathcal{S}_t$ is regular if equality holds in Inequality (8), hence $|\mathcal{S}_t|$ is maximal.
If $P \in \mathcal{S}_t$, then there are t tangents through P. If $R \notin \mathcal{S}_t$, then there are n tangents through R.

Hence the tangents form a set of type (t, n) in the dual plane.
There are several conditions (remember *Vito's* talk yesterday!), e.g.

$$\implies n - t | q.$$

The small planes

Theorem

The spectrum of the sizes of semiovals in $PG(2,q)$ *is the following:*

- *If* $q = 3$ *then* $|\mathcal{S}_1| \in \{4, 6\}$.
- *If* $q = 5$ *then* $|\mathcal{S}_1| \in \{6, 8, 9, 10, 11, 12\}$.
- *If* $q = 7$ *then* $|\mathcal{S}_1| \in \{8, 9, 12, 13, 14, 15, 16, 17, 18, 19\}$.
- *If* $q = 9$ *then* $|\mathcal{S}_1| \in \{10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28\}$.

Theorem

- *In* $PG(2,11)$ *there are semiovals of size* $12, 15, 20,$ *and for each* s *satisfying* $22 \leq s \leq 34$.
- *In* $PG(2,13)$ *there are semiovals of size* $14, 18, 24,$ *and for each integer* s *satisfying* $26 \leq s \leq 40$.

The small planes

Theorem

- *In* $\mathrm{PG}(2,2)$ *each 2-semiarc* $\mathcal{S}_2$ *consists of two or three collinear points.*
- *In* $\mathrm{PG}(2,3)$ *each 2-semiarc* $\mathcal{S}_2$ *is a set of 3 non-collinear points.*

Theorem

In $\mathrm{PG}(2,4)$ *there are three projectively non-equivalent 2-semiarcs.*

- $|\mathcal{S}_2| = 4$, *four points in general position.*
- $|\mathcal{S}_2| = 6$, *the vertices of a complete quadrilateral.*
- $|\mathcal{S}_2| = 7$, *the points of a subplane of order 2.*

$\mathrm{PG}(2,5)$

Theorem

In $\mathrm{PG}(2,5)$ there are three projectively non-equivalent 2-semiarcs.

- $|\mathcal{S}_2| = 5$, *five points of a conic.*
- $|\mathcal{S}_2| = 6$, *the union of two trisecant s.*
- $|\mathcal{S}_2| = 9$, *the projective triangle.*

PG(2, 7)

Theorem

In $\mathrm{PG}(2,7)$ *there are nine combinatorially non-equivalent 2-semiarcs (there are projectively non-equivalent subclasses in some combinatorial classes).*

$\mathrm{PG}(2,7)$

Theorem

- $|\mathcal{S}_2| = 7$, *seven points of a conic.*
- $|\mathcal{S}_2| = 9$, *there are two types,*
  1. *nine vertices of a* 3×3 *grid,*
  2. *the six vertices of two triangles* $\mathcal{C}_1$ *and* $\mathcal{C}_2$, *and the three points of intersections of the corresponding sides of* $\mathcal{C}_1$ *and* $\mathcal{C}_2$.
- $|\mathcal{S}_2| = 10$, *there are two types,*
  1. *the union of two 5-secants,*
  2. *the points of a* 10_3 *configuration.*
- $|\mathcal{S}_2| = 11$, *there is no 5-secant, there are two types,*
  1. *four 4-secants and four trisecants,*
  2. *one 4-secant and ten trisecant s.*
- $|\mathcal{S}_2| = 12$, *then it has three 4-secants, there are two types*
  1. *the 4-secants form a triangle,*
  2. *one of the 4-secant is intersected by the two others in distinct points.*

Semiarcs contained in the union of two lines

Proposition

If a t-semiarc $\mathcal{S}_t$ *is contained in the union of two lines* ℓ_1 *and* ℓ_2 *of* Π_q *and* $1 \leq t < q-1$*, then* $|\mathcal{S}_t \cap \ell_i| = q - t$ *for* $i = 1, 2$*, and* $\mathcal{S}_t$ *does not contain the point* $\ell_1 \cap \ell_2$*.*

Semiovals contained in the union of three lines

Theorem

Let $\mathcal{S}_1$ be a semioval in a projective plane Π_q. If $\mathcal{S}_1$ is contained in the union of three lines then

$$\frac{3(q-1)}{2} \leq |S| \leq 3(q-1).$$

Theorem (Gy. K. and J. Ruff, 2006)

A semioval in $PG(2,q)$ which is contained in the sides of a triangle and which contains one vertex of this triangle has a $(q-2)$-secant and two $(t+1)$-secants where t is a suitable integer. This type of semiovals exists if and only if $q=4$ and $t=1$, $q=8$ and $t=4$ or $q=32$ and $t=26$.

Semiovals contained in the union of three lines

Theorem (Gy. K. and J. Ruff, 2006)

If a semioval S in $PG(2,q)$ is contained in the sides of a triangle $\mathcal{T}$ and does not contain any vertex of $\mathcal{T}$, then there are two possibilities:

- *$\mathcal{S}_1$ has two $(q-1)$-secants and a k-secant. Semiovals in this class exist for all $1 < k < q$.*
- *$\mathcal{S}_1$ has three $(q-1-d)$-secants where d is a suitable divisor of $q-1$.*

Three concurrent lines

Theorem

If a semioval $\mathcal{S}_1$ *in* $PG(2,q)$ *is contained in the union of three concurrent lines then* $|S| > 3(q-1)/2$ *for* $q > 9$.

Definition

Let ℓ_1, ℓ_2 and ℓ_3 be the three concurrent lines whose union contains $\mathcal{S}$. We denote by C the common point of these three lines and by $\mathcal{L}$ the union of ℓ_1, ℓ_2 and ℓ_3. And finally, we let $\mathcal{L}_i = \mathcal{S} \cap \ell_i$ $(i = 1,2,3)$. The semioval $\mathcal{S}$ is *strong*, if for any point $K \in \mathcal{L} \setminus (\mathcal{S} \cup \{C\})$, the number of two-secants of $\mathcal{S}$ passing through K is independent of K.

Semiarcs contained in three concurrent lines

ℓ_1, ℓ_2 and ℓ_3 the three concurrent lines whose union contains $\mathcal{S}_t$,
$V = \ell_1 \cap \ell_2 \cap \ell_3$, $\mathcal{P}_V$: pencil of lines with carrier V,
$\mathcal{L}_i = \mathcal{S}_t \cap \ell_i$, $u_i = |\mathcal{L}_i|$.

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Proposition

Let $\mathcal{S}_t$ be a t-semiarc in Π_q, suppose that $\mathcal{S}_t$ is contained in the union of three lines of $\mathcal{P}_V$, but does not contained in the union of any two lines of $\mathcal{P}_V$. If $V \in \mathcal{S}_t$, then there are two possibilities.

- *$\mathcal{S}_t$ consists of the six vertices of a complete quadrilateral,*
- *$\mathcal{S}_t$ is a Fano subplane.*

In both cases $t = q - 2$.

Three concurrent lines II

Theorem ($V \notin \mathcal{S}_t$)

Let $\mathcal{S}_t$ be a t-semiarc in Π_q, suppose that $\mathcal{S}_t$ is contained in the union of three lines of $\mathcal{P}_V$, but does not contained in the union of any two lines of $\mathcal{P}_V$. If $V \notin \mathcal{S}_t$, then there are three possibilities.

1. $u_1 = u_2 = u_3 = u$, *and*

$$3 \cdot \frac{q-t}{2} \leq |\mathcal{S}_t| \leq 3 \cdot \left(q + \frac{t}{2} - \sqrt{qt + \frac{t^2}{4}} \right).$$

2. $u_i = u_j = q - t$ *and* $2 \leq u_k \leq t$ *holds for* $\{i, j, k\} = \{1, 2, 3\}$. *The inequalities*

$$2q - 2t + 2 \leq |\mathcal{S}_t| \leq 2q - t \tag{1}$$

also hold in this case.

3. $\mathcal{S}_t$ *is a 5-arc and* $t = q - 3$.

Again an example

If Π_q contains a sublane of order t, then there exist t-semiarcs of each possible size allowed by Inequality (1).

Example

Suppose that Π_q contains a subplane Π_t of order t. Let ℓ_1, ℓ_2 and ℓ_3 be lines of Π_t. Let $Z \subseteq (\ell_3 \cap \Pi_t) \setminus \{V\}$ an arbitrary subset satisfying the inequalities $2 \leq |Z| \leq t$. Finally let

$$\mathcal{S}_t := (\ell_1 \setminus \Pi_t) \cup (\ell_2 \setminus \Pi_t) \cup Z.$$

$\mathcal{S}_t \iff$ ordered triple (A, B, C),
where $A, B, C \subset \mathrm{GF}(q)$.
The lines ℓ_1, ℓ_2, and ℓ_3 have equations $X_1 = 0$, $X_1 = X_3$ and $X_3 = 0$, respectively. $V = (0, 1, 0)$.

$$A = \{a \in \mathrm{GF}(q) : (0, a, 1) \notin \mathcal{L}_1\},$$

$$B = \{b \in \mathrm{GF}(q) : (1, b, 1) \notin \mathcal{L}_2\},$$

$$C = \{c \in \mathrm{GF}(q) : (1, c, 0) \notin \mathcal{L}_3\}.$$

PG(2, q), algebraic description

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$$(0, a, 1), (1, b, 1), (1, c, 0) \text{ collinear } \iff a + c = b.$$

Triangular case

The line ℓ_i has equation $X_i = 0$.

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$$(0, a, 1),\ (b, 0, 1),\ (1, c, 0)\,\text{collinear} \iff ac = -b.$$

Definitions from Additive Group Theory

Definition

Let A and B be finite, nonempty subsets of an abelian group $(Z, \odot)$, and let $i \geq 1$ an integer.

- Let $N_i(A, B)$ all the elements c with at least i representations of the form $c = a \odot b$ with $a \in A$ and $b \in B$. Sometimes we use the shorthand notation N_i instead of $N_i(A, B)$.
- Let $stab(A) := \{z \in Z \ : A \odot z = A\}$. This set is called the stabilizer of A.

Theorems from Additive Group Theory

Theorem (Exact inverse sumset theorem)

Suppose that A and B are finite nonempty subsets of the abelian group Z. Then the following are equivalent.

- $|A + B| = |A|$.
- $|A - B| = |A|$.
- *Let $G := stab(A)$. Then G is a finite subgroup of Z, B is contained in a coset of G, and A is the union of cosets of of G.*

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Theorem (Pollard, 1974)

Let Z be an abelian group, $|Z| = p$ prime, $A, B \subseteq G$ nonempty subsets, and $1 \leq k \leq min\{|A|, |B|\}$. Then

$$|N_1| + |N_2| + \ldots + |N_k| \geq k \cdot min\{p, |A| + |B| - k\}.$$

Theorem (Grynkiewicz, 2010)

Let Z be an abelian group, $A, B \subseteq Z$ finite and nonempty subsets, and $k \geq 1$. If $|A|, |B| \geq k$, then either

$$\sum_{i=1}^{k} |N_i| \geq k(|A| + |B|) - 2k^2 + 1,$$

Theorem

or else there exist $A' \subseteq A$ *and* $B' \subseteq B$ *with*

$$l := \left|A \setminus A'\right| + \left|B \setminus B'\right| \leq k - 1,$$

$$N_k(A', B') = N_1(A', B') = N_k(A, B),$$

$$\sum_{i=1}^{k} |N_k| \geq k(|A| + |B|) - (k - l)(|H| - \rho) - kl \geq k(|A| + |B| - |H|),$$

where H *is the nontrivial stabilizer of* $N_k(A, B)$ *and* $\rho = |A' \odot H| - |A'| + |B' \odot H| - |B'|$. *In the case* $k = 2$ *instead of the first inequality* $|N_1| + |N_2| \geq 2(|A| + |B|) - 4$ *also holds.*

Application of the Inverse Sumset Thm

Theorem

Suppose that the t-semiarc $\mathcal{S}_t$ in $PG(2, p^r)$, p odd prime, belongs to the family of Case 2 of Theorem $V \notin \mathcal{S}_t$. Then there exists a subgroup G of E such that both A and C are union of cosets of G, and $\overline{B}$ is contained in a coset of G.
If ϕ is the natural homomorphism from E to E/G, $|G| = g$ and $|\phi(C)| = h$, then $t = gh$ and $|\mathcal{S}_t| = 2p^r - 2gh + |\overline{B}|$.

Corollary

Let p be an odd prime. Then the followings hold.

1. *In $PG(2, p)$ there is no semiarc belonging to the family of Case 2 of Theorem $V \notin \mathcal{S}_t$.*
2. *Let $1 \leq e < r$ be integers and let $t = p^e s$, where $(p, s) = 1$ and $t < p^r$. Then $PG(2, p^r)$ contains t-semiarcs with cardinality $2p^r - 2t + k$ for all t and k satisfying the conditions $2 \leq k \leq p^e$.*

Theorem

Let $\mathcal{S}_1$ be a semioval in the plane $PG(2,q)$, $q = p^r$, p odd prime. Suppose that $\mathcal{S}_1$ is contained in the union of three lines of $\mathcal{P}_V$, but does not contained in the union of any two lines of $\mathcal{P}_V$. Then $|\mathcal{S}_1| \geq 3q - 3f_r(q)$, where

$$f_r(q) = \begin{cases} 2\lceil\sqrt{p+1}\rceil - 2 & \textit{if } r = 1, \\ 4\left\lceil\sqrt{\frac{q+1}{2}}\right\rceil - 4 & \textit{if } r = 2, \\ q^{\frac{r-1}{r}} + q^{\frac{1}{r}} - 1 & \textit{if } r \geq 3. \end{cases}$$

Strong semiovals

Theorem (B. Csajbõk, Gy. K., 2012)

Let $\mathcal{S}_1$ be a strong semioval in $PG(2, p^r)$, p an odd prime. Then the followings hold.

1. *If $r = 2l$, then $\mathcal{S}_1$ contains $3(p^{2l} - p^l)$ points.*
2. *If $r = 2l + 1$ and $p > 7$, then there is no strong semioval in $PG(2, p^r)$.*
3. *If $r = 2l + 1$ and $p = 3, 5$ or 7, then $\mathcal{S}_1$ contains $3(p^{2l+1} - p^{l+1})$ points.*

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2. *If $r = 2l + 1$ and $p > 7$, then there is no strong semioval in $PG(2, p^r)$.*
3. *If $r = 2l + 1$ and $p = 3, 5$ or 7, then $\mathcal{S}_1$ contains $3(p^{2l+1} - p^{l+1})$ points.*

Conjecture

The projective plane $PG(2, q)$, q odd, contains strong semiovals if and only if q is a square.

Theorem

Let $\mathcal{S}_2$ be a 2-semiarc in $PG(2,q)$, $q = p^r$, p odd prime. Suppose that $\mathcal{S}_2$ belongs to the family of Case 1 of Theorem $V \notin \mathcal{S}_t$. Then $|\mathcal{S}_2| \geq 3q - 3f_r(p)$, where

$$f_r(p) = \begin{cases} 2\lceil\sqrt{2p+4}\rceil - 4 & \textit{if } r = 1, \\ 4\left\lceil\sqrt{p^2 + \frac{7}{2}}\right\rceil - 8 & \textit{if } r = 2, \\ 14, 37, 66 & \textit{if } r = 3 \textit{ and } p = 3,\ 5,\ 7, \\ p^2 + 2p + 2 & \textit{if } r = 3 \textit{ and } p \geq 11, \\ p^{r-1} + 2p - 2 & \textit{if } r \geq 4. \end{cases}$$

The next example

Example

Let E be the direct sum of the subgroups X and Y. Then the sets $A = B = C := X \cup Y$ define a 2-semiarc $\mathcal{S}_2$. This semiarc has $3q - 3(|X| + |Y| - 1)$ points, hence if $q = p^r$ and $|X| = p^{r-1}$, $|Y| = p$, then $|\mathcal{S}_2| = 3q - 3(p^{r-1} + p - 1)$.

Semiarcs with long secants

Proposition

Let $\mathcal{S}_t$ be a t-semiarc and ℓ be a line in Π_q. Suppose that $|\mathcal{S}_t \cap \ell| = k$. Then

$$k \leq q + 1 - t, \tag{2}$$

$$k + q - t \leq |\mathcal{S}_t| \leq k + \frac{(q + 1 - k)q}{t}. \tag{3}$$

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$$k + q - t \leq |\mathcal{S}_t| \leq k + \frac{(q+1-k)q}{t}. \tag{3}$$

These inequalities are sharp.

Long secants

Theorem (B. Csajbók)

In $\mathrm{PG}(2,q)$*, a* t*-semiarc with a* $(q+1-t)$*-secant exists if and only if* $t \geq (q-1)/2$*.*

Proposition (no. 44)

Let Π_q *be a projective plane of order* q*. If* $\mathcal{S}_t$ *is a* t*-semiarc of size* $2(q-t)$ *with a* $(q-t)$*-secant* ℓ*, then* $\mathcal{S}_t$ *consists of the symmetric difference of two lines, with* t *further points removed from each line.*

Semiarcs with two long secants

Lemma (B. Csajbók, 2012)

Let Π_q be a projective plane of order q, $1 < t < q$ an integer and $\mathcal{S}_t$ be a t-semiarc in Π_q. Suppose that there exist two lines ℓ_1 and ℓ_2 such that $|\ell_1 \setminus (\mathcal{S}_t \cup \ell_2)| = n$ and $|\ell_1 \setminus (\mathcal{S}_t \cup \ell_2)| = m$. If $\ell_1 \cap \ell_2 \notin \mathcal{S}_t$, then either $n = m = t$ or $q \leq n + 2nm/(t-1)$.

t-semiarcs in $\mathrm{PG}(2, q)$ with two $(q-t)$-secants

If ℓ_1 and ℓ_2 are $(q-t)$-secants, then each point in $\mathcal{S}_t \setminus (\ell_1 \cup \ell_2)$ is a center of a perspectivity mapping $\ell_1 \setminus \mathcal{S}_t$ to $\ell_2 \setminus \mathcal{S}_t$. The description of perspective pointsets was given by Korchmáros and Mazzocca. Applying their result Csajbók proved the folliwing.

Theorem (B. Csajbók, 2012)

Let $\mathcal{S}_t$ be a t-semiarc in $\mathrm{PG}(2, q)$, $q = p^h$, p prime, with two $(q-t)$-secants such that the point of intersection of these secants is not contained in $\mathcal{S}_t$. Then the following hold.

1. *If $\gcd(q, t) = 1$, then $\mathcal{S}_t$ is contained in a vertexless triangle.*
2. *If $\gcd(q, t) = 1$ and $\gcd(q-1, t-1) = 1$, then $\mathcal{S}_t$ consists of the symmetric difference of two lines, with t further points removed from each line.*
3. *If $\gcd(q-1, t) = 1$, then $\mathcal{S}_t$ is contained in a vertexless triangle, or in the union of three concurrent lines with their common point removed.*

Small semiarcs with long secants

The following example shows the existence of t-semiarcs of size $k+q-t$ with three k-secants for odd values of t.

Example (no. 47)

Let C denote the set of non-squares in $\mathrm{GF}(q)$, q odd. The pointset $\{(0:1:s),(s:0:1),(1:s:0)\ :\ -s \in C\}$ is a semioval in $\mathrm{PG}(2,q)$ of size $3(q-1)/2$ with three $(q-1)/2$-secants. If we delete $r < (q-1)/2 - 1$ points from each of the $(q-1)/2$-secants, then the remaining pointset is a t-semiarc of size $k+q-t$ with three k-secants, where $k = (q-1)/2 - r$ and $t = 2r+1$.

Long secants, t even

A $(q+t,\ t)$-arc of type $(0,\ 2,\ t)$, $t \neq 0, 2$, in Π_q is a set $\mathcal{T}$ of $q+t$ points in $\mathrm{PG}(2,q)$ for which each line ℓ meets $\mathcal{T}$ in 0, 2 or t points. Korchmáros and Mazzocca proved that $(q+t,\ t)$-arcs of type $(0,\ 2,\ t)$ exist in $\mathrm{PG}(2,q)$ only if q is even and $t|q$.

Example

Let $\mathcal{T}$ be a $(q+\tau,\ \tau)$-arc of type $(0,\ 2,\ \tau)$ in $\mathrm{PG}(2,q)$. Delete $r < \tau - 1$ points from each of the τ-secants of $\mathcal{T}$. The remaining $k+q-t$ points form a t-semiarc with $q/\tau + 1$ k-secants, where $k = \tau - r$ and $t = rq/\tau$.

Example (no. 49)

Let $\Pi_{\sqrt{q}}$ be a Baer subplane in the projective plane Π_q, $q \geq 9$, and let ℓ be a line of $\Pi_{\sqrt{q}}$. Let $\mathcal{P}$ be a set of t points in $\Pi_{\sqrt{q}} \setminus \ell$ such that no line intersects $\mathcal{P}$ in exactly $\sqrt{q}-1$ points. For example a $(t, \sqrt{q}-2)$-arc is a good choice for $\mathcal{P}$. Let $\mathcal{T}$ be a set of t points in $\ell \setminus \Pi_{\sqrt{q}}$. Then the pointset $\mathcal{S}_t := (\Pi_{\sqrt{q}} \triangle \ell) \setminus (\mathcal{T} \cup \mathcal{P})$ is a t-semiarc of size $k+q-t$ with a k-secant, where $k = q - \sqrt{q} - t$.

Long secants and the direction problem

The so-called direction problem is closely related to t-semiarcs of size $k+q-t$ having a k-secant. Consider $\mathrm{PG}(2,q) = \mathrm{AG}(2,q) \cup \ell_\infty$. Let $\mathcal{U}$ be a set of points in $\mathrm{AG}(2,q)$. A point P of ℓ_∞ is called a direction determined by $\mathcal{U}$ if there is a line through P that contains at least two points of $\mathcal{U}$.

Theorem

Let $\mathcal{S}_t$ be a t-semiarc in $\mathrm{PG}(2,q)$ $q = p^h$, p prime, of size $k+q-t$ and let ℓ be a k-secant of $\mathcal{S}_t$. Then the conditions

- $t = 1$, $q > 4$ *and* $k > (q-1)/2$, *or*
- $2 \leq t \leq \alpha\sqrt{q}$ *and* $k > \alpha(q+1)$ *for some* $1/2 \leq \alpha \leq \sqrt{(p-1)/p}$ *if p is an odd prime and* $1/2 \leq \alpha \leq \sqrt{3}/2$ *if* $p = 2$

imply that $\mathcal{S}_t$ is either a semiarc described in Example 49, or $k = q-t$ and $\mathcal{S}_t$ is the semiarc described in Proposition 44.

Long secants

Corollary

Let $\mathcal{S}_1$ be a semioval of size $k+q-1$ in $\mathrm{PG}(2,q)$, $q=p^h$, p prime, $h \leq 2$, and let ℓ be a k-secant of $\mathcal{S}_1$. Then we have the following.

1. *If $h=1$ and $k>(p+4)/3$, then there are two possibilities:*
   - *$k=q-1$ and $\mathcal{S}_1$ is the semioval described in Proposition 44,*
   - *$\mathcal{S}_1$ is the semioval described in Example 47.*
2. *If $h=2$ and $k>(p^2-p)/2$, then there are four possibilities:*
   - *$k=q-1$ and $\mathcal{S}_1$ is the semioval described in Proposition 44,*
   - *$\mathcal{S}_1$ is the semioval described in Example 47,*
   - *$\mathcal{S}_1$ is the semioval described in Example 49,*
   - *$p=2$ and $\mathcal{S}_1$ is an oval in $\mathrm{PG}(2,4)$.*

Semiovals with a long secants

Theorem (Gy. K., 2004)

Let $\mathcal{S}_1$ be a semioval in the Desarguesian plane $PG(2,q)$. If there exist integers $1 \leq r$ and $-1 \leq k$ such that $|\mathcal{S}_1| = 2q - r + k$, $r + 4k + 4 < q$, $2(r + k) < q$ and $\mathcal{S}_1$ has a $(q - r)$-secant, then the tangent lines at the points of the $(q - r)$-secant are concurrent.

A much more general result

Lemma (B. Csajbók, T. Héger, Gy. K., 2013)

Let $\mathcal{S}_t$ be a set of points in $\mathrm{PG}(2,q)$, let ℓ be a k-secant of $\mathcal{S}_t$ with $k \leq q$ and let $1 \leq t \leq q-3$ be an integer. Suppose that through each point of $\ell \cap \mathcal{S}_t$ there pass exactly t tangent lines to $\mathcal{S}_t$. Denote by s the size of $\mathcal{S}_t$ and let $s = k + q - t + \epsilon$. Let $A(n)$ be the set of those points in $\ell \setminus \mathcal{S}_t$ through which there pass at most n skew lines of $\mathcal{S}_t$. Then the following hold.

A much more general result

Lemma

- *If $t = 1$ and*
  1. $\epsilon < \frac{k}{2} - 1$, *then the k tangent lines at the points of $\mathcal{S}_1 \cap \ell$ and the skew lines through the points of $A(2)$ belong to a pencil (hence $A(2) \setminus A(1)$ is empty),*
  2. *if $\epsilon < \frac{2k}{3} - 2$, then the k tangent lines at the points of $\mathcal{S}_1 \cap \ell$ either belong to two pencils or they form a dual k-arc. If $k < q$, then the skew lines through the points of $A(2)$ belong to the same pencils or dual k-arc.*
- *If $t \geq 2$ and $k > q - \frac{q}{t} + 1$, then*
  3. *if $\epsilon < \frac{k}{t+1} - \frac{t}{2}$, then the kt tangent lines at the points of $\mathcal{S}_1 \cap \ell$ and the skew lines through the points of $A(t+1)$ belong to t pencils whose carriers are not on ℓ (hence $A(t+1) \setminus A(t)$ is empty),*
  4. *if $\epsilon < \frac{k}{t+1} - 1$ and $t \leq \sqrt{q}$, then the kt tangent lines at the points of $\mathcal{S}_1 \cap \ell$ belong to $t+1$ pencils whose carriers are not on ℓ. If $k < q$, then the skew lines through the points of $A(t+1)$ belong to the same pencils.*

Some corollaries

Corollary

Let $\mathcal{S}_1$ *be a semioval in* $\mathrm{PG}(2,q)$ *and let* ℓ *be a* k*-secant of* $\mathcal{S}_1$*. If* $|\mathcal{S}_1| < q + \frac{3k}{2} - 2$*, then the* k *tangent lines at the points of* $\mathcal{S}_1 \cap \ell$ *belong to a pencil. If* $|\mathcal{S}_1| < q + \frac{5k}{3} - 3$*, then the* k *tangent lines at the points of* $\mathcal{S}_1 \cap \ell$ *either belong to two pencils or they form a dual* k*-arc.*

Lemma

Let $k \leq q$ and $1 \leq t \leq q-3$ be integers. Let $\mathcal{S}_t$ be a set of $k+q-t+\epsilon$ points in Π_q such that the line ℓ is a k-secant of $\mathcal{S}_t$. Let $A(n)$ be the set of those points in $\ell \setminus \mathcal{S}_t$ through which there pass at most n skew lines of $\mathcal{S}_t$. Suppose that through each of the k points of $\ell \cap \mathcal{S}_t$ there pass exactly t tangent lines to $\mathcal{S}_t$, and also suppose that these kt tangent lines and the skew lines through the points of $A(n)$ belong to n pencils. Let $\mathcal{P}$ be the set of carriers of these pencils and assume that $\mathcal{P} \cap \ell = \emptyset$. Define the pointset $\mathcal{B}_n(\mathcal{S}_t, \ell)$ in the following way:

$$\mathcal{B}_n(\mathcal{S}_t, \ell) := (\ell \setminus (A(n) \cup \mathcal{S}_t)) \cup (\mathcal{S}_t \setminus \ell) \cup \mathcal{P}.$$

Then $\mathcal{B}_n(\mathcal{S}_t, \ell)$ has size $2q+1+\epsilon+n-t-k-|A(n)|$ and it is an affine blocking set in the affine plane $\Pi_q \setminus \ell$ or a blocking set in Π_q. In the latter case the points of $\ell \cap \mathcal{B}_n(\mathcal{S}_t, \ell)$ are essential points.

Lemma

Let $k \leq q$ and $1 \leq t \leq q-3$ be integers. Let $\mathcal{S}_t$ be a set of $k+q-t+\epsilon$ points in Π_q such that the line ℓ is a k-secant of $\mathcal{S}_t$. Let $A(n)$ be the set of those points in $\ell \setminus \mathcal{S}_t$ through which there pass at most n skew lines of $\mathcal{S}_t$. Suppose that through each of the k points of $\ell \cap \mathcal{S}_t$ there pass exactly t tangent lines to $\mathcal{S}_t$, and also suppose that these kt tangent lines and the skew lines through the points of $A(n)$ belong to n pencils. Let $\mathcal{P}$ be the set of carriers of these pencils and assume that $\mathcal{P} \cap \ell = \emptyset$. Define the pointset $\mathcal{B}_n(\mathcal{S}_t, \ell)$ in the following way:

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Then $\mathcal{B}_n(\mathcal{S}_t, \ell)$ has size $2q+1+\epsilon+n-t-k-|A(n)|$ and it is an affine blocking set in the affine plane $\Pi_q \setminus \ell$ or a blocking set in Π_q. In the latter case the points of $\ell \cap \mathcal{B}_n(\mathcal{S}_t, \ell)$ are essential points.

Theorem

Let $\mathcal{S}_t$ be a t-semiarc in $\mathrm{PG}(2,p)$, p prime, and let ℓ be a k-secant of $\mathcal{S}_t$.

1. *If $t = 1$, $p \geq 5$ and $k \geq \frac{p-1}{2}$, then*
   - *$\mathcal{S}_1$ is contained in a vertexless triangle and has two $(p-1)$-secants, or*
   - *$\mathcal{S}_1$ is projectively equivalent to Example 47, or*
   - $|\mathcal{S}_1| \geq \min\{\frac{3k}{2} + p - 2, 2k + \frac{p+1}{2}\}$.
2. *If $t = 2$, $p \geq 7$ and $k \geq \frac{p+3}{2}$, then*
   - *$\mathcal{S}_2$ consists of the symmetric difference of two lines, with two further points removed from each line, or*
   - $|\mathcal{S}_2| \geq \min\{\frac{4k}{3} + p - 3, 2k + \frac{p-1}{2}\}$.
3. *If $3 \leq t < \sqrt{p}$, $p \geq 23$ and $k > p - \frac{p}{t} + 1$, then*
   - *$\mathcal{S}_t$ is contained in a vertexless triangle and has two $(p-t)$-secants, or*
   - $|\mathcal{S}_t| \geq k\frac{t+2}{t+1} + p - t - 1$.

Theorem

Let $\mathcal{S}_t$ be a t-semiarc in $\mathrm{PG}(2,q)$, $q = p^h$, $h > 1$ if p is an odd prime and $h \geq 6$ if $p = 2$. Let the size of the smallest minimal non-trivial blocking set in $\mathrm{PG}(2,q)$ be denoted by $q + 1 + b(q)$, and suppose that $\mathcal{S}_t$ has a k-secant ℓ with $k \geq q - b(q) - t$. Then the following hold.

1. *If $h = 2d$, $|\mathcal{S}_t| \leq 2k + b(q)$ and $t < \Phi(\sqrt{q} - 1)$, then*
   - *$\mathcal{S}_t$ has two $(q-t)$-secants, having no common point in $\mathcal{S}_t$, or*
   - *$\mathcal{S}_t$ is as in Example 49.*
2. *If $h = 2d + 1$, $|\mathcal{S}_t| < 2k + b(q)$ and $t < q^{1/3} - 3/2$ (or $t < (2q)^{1/3} - 2$, when $p = 2, 3$), then $\mathcal{S}_t$ has two $(q-t)$-secants whose common point is not in $\mathcal{S}_t$.*

THANK YOU FOR YOUR ATTENTION!