Valuations on Convex Bodies and Sobolev Spaces

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Szeged, May 2012

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- $\langle \mathbb{A}, + \rangle$ Abelian semigroup

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$$\mathrm{z}(K) + \mathrm{z}(L) = \mathrm{z}(K \cup L) + \mathrm{z}(K \cap L)$$

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- Hilbert's Third Problem:
Dehn 1902, Sydler 1965, Jessen & Thorup 1978, McMullen 1989, ...
- Classification of valuations:

Blaschke 1937, **Hadwiger** 1949, Schneider 1971, Groemer 1972, McMullen 1977, Betke & Kneser 1985, Klain 1995, L. 1999, Reitzner 1999, Alesker 1999, Fu 2006, Haberl 2006, Schuster 2006, Tsang 2010, Wannerer 2010, Abardia 2011, Bernig & Fu 2011, Parapatits 2011, ...

Hadwiger's Classification Theorem 1952

Theorem

A functional $\mathrm{z} : \mathcal{K}^n \to \langle \mathbb{R}, + \rangle$ *is a continuous and rigid motion invariant valuation*

$$\iff$$

$\exists\, c_0, c_1, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{z}(K) = c_0\, V_0(K) + \cdots + c_n\, V_n(K)$$

for every $K \in \mathcal{K}^n$.

$V_0(K), \dots, V_n(K)$	intrinsic volumes of K
V_n	n-dimensional volume
$n\, V_{n-1}(K) = S(K)$	surface area

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New proof by Dan Klain 1995

"Introduction to Geometric Probability" by Klain & Rota 1997

Valuations on Convex Bodies

- Real valued valuations $\mathrm{z} : \mathcal{K}^n \to \langle \mathbb{R}, + \rangle$:
  - Translation invariant, continuous: McMullen, Alesker, ...
  - Rotation invariant, continuous: Alesker, ...
  - $\mathrm{SL}(n)$ invariant, upper semicontinuous: L. & Reitzner, ...

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- Tensor valued valuations $\mathrm{z}: \mathcal{K}^n \to \langle \mathbb{T}^d, + \rangle$:
  - Vector valued: Hadwiger & Schneider, ...
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- Body valued valuations:
  - Minkowski valuations $\mathrm{z} : \mathcal{K}^n \to \langle \mathcal{K}^n, + \rangle$:
    L., Schuster, Schneider, Wannerer, Abardia & Bernig, ...
  - L^p Minkowski valuations $\mathrm{z} : \mathcal{K}^n \to \langle \mathcal{K}^n, +_p \rangle$:
    L., Haberl, Wannerer, Parapatits, ...
  - Radial valuations $\mathrm{z} : \mathcal{K}^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$: L., Haberl, ...
  - Blaschke valuations $\mathrm{z} : \mathcal{K}^n \to \langle \mathcal{K}^n_c, \# \rangle$: Haberl, ...

Valuations on Function Spaces

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- $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$

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- Valuations on L^p stars: Dan Klain (AIM 1996, 1997)
- Valuations on L^p spaces: Andy Tsang (IMRN 2010, TAMS 2011+)
- Valuations on Orlicz spaces: Hassane Kone (2011+)
- Valuations on $\mathrm{BV}(\mathbb{R}^n)$: Tuo Wang (2011+)
- Valuations on Sobolev spaces: L. (AIM 2011, AJM 2012, ...)

Sobolev Spaces

- $W^{1,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : |\nabla f| \in L^p(\mathbb{R}^n)\}$
  $p \geqslant 1,\ n \geqslant 3$

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Question

Classification of valuations on $W^{1,p}(\mathbb{R}^n)$

Matrix-Valued Valuations on Sobolev Spaces

- $W^{1,2}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : |\nabla f| \in L^2(\mathbb{R}^n)\}$

- $\mathbb{M}^n$ space of symmetric $n \times n$ matrices

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- $\mathrm{Z} : W^{1,2}(\mathbb{R}^n) \to \mathbb{M}^n$ is $\mathrm{GL}(n)$ contravariant $\Leftrightarrow$

$$\mathrm{Z}(f \circ \phi) = |\det \phi|^q \, \phi^{-t} \, \mathrm{Z}(f) \, \phi^{-1} \qquad \forall \, \phi \in \mathrm{GL}(n)$$

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- $\mathrm{Z} : W^{1,2}(\mathbb{R}^n) \to \mathbb{M}^n$ is **affinely contravariant** $\Leftrightarrow$
  Z is $\mathrm{GL}(n)$ contravariant, translation invariant and homogeneous

A Characterization of the Fisher Information Matrix

Theorem (L.: AIM 2011)

A function $\mathrm{Z} : W^{1,2}(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$ *is a continuous and affinely contravariant valuation*

$$\iff$$

$\exists\, c \in \mathbb{R}$ *such that*

$$\mathrm{Z}(f) = c\, \mathrm{J}(f^2)$$

for every $f \in W^{1,2}(\mathbb{R}^n)$.

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- $$\mathrm{J}_{ij}(g) = \int_{\mathbb{R}^n} \frac{\partial \log g(x)}{\partial x_i} \frac{\partial \log g(x)}{\partial x_j}\, g(x)\, dx$$
- Connection between Fisher information matrix and LYZ ellipsoid (Lutwak, Yang & Zhang: DMJ 2000)

A Characterization of the LYZ Matrix

Theorem (L.: DMJ 2003)

A function $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathbb{M}, + \rangle$ *is a valuation such that*

$$\mathrm{Z}(\phi P) = |\det \phi|\, \phi^{-t}\, \mathrm{Z}(P) \phi^{-1} \quad \forall\, \phi \in \mathrm{GL}(n)$$

$$\Longleftrightarrow$$

$\exists\, c \in \mathbb{R}$ *such that*

$$\mathrm{Z}(P) = c\, \mathrm{L}(P)$$

for every $P \in \mathcal{P}_0^n$.

- $\mathcal{P}_0^n$ convex polytopes in $\mathbb{R}^n$ containing the origin in their interiors
- $\mathrm{L}_{ij}(P) = \sum_u \frac{a(P,u)}{h(P,u)}\, u_i\, u_j$
  - u facet normal
  - $a(P,u)$ facet area
  - $h(P,u)$ support distance

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A Characterization of the LYZ operator

Theorem (L.: AJM 2012)

An operator $\mathrm{Z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle$ *is a continuous and affinely contravariant valuation*

$$\iff$$

$\exists\, c \geqslant 0$ *such that*

$$\mathrm{Z}(f) = c\, \Pi \langle f \rangle$$

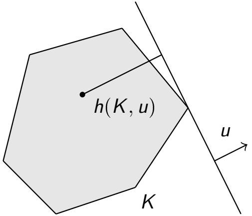
for every $f \in W^{1,1}(\mathbb{R}^n)$.

- $K + L = \{x + y : x \in K, y \in L\}$ Minkowski sum of $K, L \in \mathcal{K}_c^n$
- $\langle \cdot \rangle : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$, $f \mapsto \langle f \rangle$ LYZ operator

  Lutwak, Yang & Zhang: IMRN 2006
- $\Pi\, K$ projection body of $K \in \mathcal{K}_c^n$

Origin-symmetric Convex Bodies $\mathcal{K}_c^n$

- Support function $h(K,\cdot):\mathbb{R}^n\to\mathbb{R}$



  - $h(K,u)=\max\{u\cdot x : x\in K\}$

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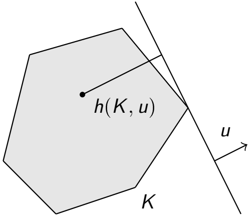

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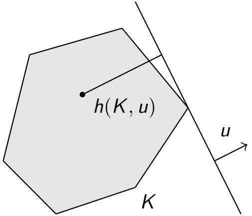

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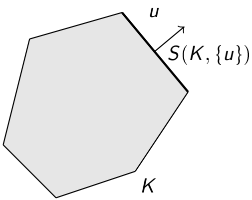
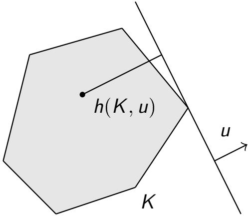
- Surface area measure $S(K,\cdot) : \mathcal{B}(\mathbb{S}^{n-1}) \to [0,\infty)$



  - $S(K,\omega) = \mathcal{H}^{n-1}(\{x \in \operatorname{bd} K : n(K,x) \in \omega\})$
  - $n(K,x)$ outer unit normal vector to K at $x \in \operatorname{bd} K$

Origin-symmetric Convex Bodies $\mathcal{K}_c^n$

- Support function $h(K,\cdot):\mathbb{R}^n \to \mathbb{R}$



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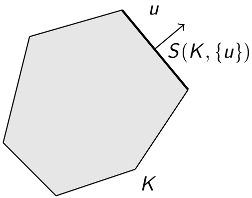
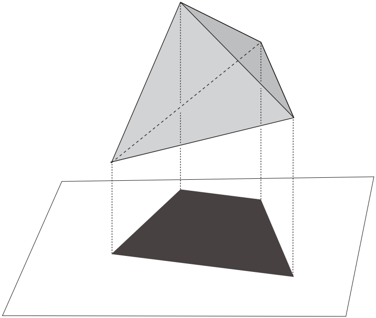


- $S(K,\omega) =$
  $\mathcal{H}^{n-1}(\{x \in \operatorname{bd} K : n(K,x) \in \omega\})$
- $n(K,x)$ outer unit normal vector to K at $x \in \operatorname{bd} K$
- $S(K,\cdot)$ even measure,
  not concentrated on a great sphere

Projection Body, ΠK, of K



- $u^\perp$ hyperplane orthogonal to u
- $K|u^\perp$ projection of K to $u^\perp$
- V_{n-1} $(n-1)$-dimensional volume

Definition (Minkowski 1901)

$$h(\Pi K, u) = V_{n-1}(K|u^\perp) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |u\cdot v|\, dS(K,v)$$

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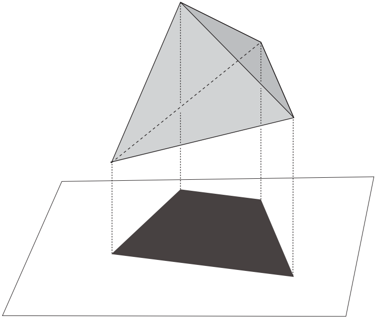

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Sobolev Inequality

$$\frac{1}{n}\int_{\mathbb{R}^n} |\nabla f(x)|\, dx \geqslant v_n^{1/n}\, |f|_{n/(n-1)}$$

- $f \in W^{1,1}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n), |\nabla f| \in L^1(\mathbb{R}^n)\}$
- $|x|$ Euclidean norm of $x \in \mathbb{R}^n$
- $|f|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx\right)^{1/p}$
- v_n volume of n-dimensional unit ball

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- v_n volume of n-dimensional unit ball
- Equality for indicator functions of balls
- Equivalent to Euclidean isoperimetric inequality
- Federer & Fleming 1960, Maz′ya 1960

General Sobolev Inequality

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\, dx \geqslant v_n^{1/n}\, |f|_{n/(n-1)}$$

- $f \in W^{1,1}(\mathbb{R}^n)$
- $\mathcal{K}_c^n$ origin-symmetric convex bodies (compact convex sets) in $\mathbb{R}^n$
- $K \in \mathcal{K}_c^n$ with $V(K) = v_n$
- $K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1 \text{ for all } y \in K\}$ polar body of K
- $\|\cdot\|_L$ norm with unit ball L

General Sobolev Inequality

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\,dx \geqslant v_n^{1/n}\,|f|_{n/(n-1)}$$

- $f \in W^{1,1}(\mathbb{R}^n)$
- $\mathcal{K}_c^n$ origin-symmetric convex bodies (compact convex sets) in $\mathbb{R}^n$
- $K \in \mathcal{K}_c^n$ with $V(K) = v_n$
- $K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1 \text{ for all } y \in K\}$ polar body of K
- $\|\cdot\|_L$ norm with unit ball L
- Equality for $f = \mathbb{1}_K$
- Equivalent to Minkowski inequality
- Gromov 1986

Optimal Sobolev Inequality

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\, dx \geqslant v_n^{1/n}\, |f|_{n/(n-1)}$$

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Question (Lutwak, Yang & Zhang 2006)

For given $f \in W^{1,1}(\mathbb{R}^n)$, which convex body K (of volume v_n) minimizes

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\, dx?$$

Which norm is optimal?

Optimal Sobolev Body

Definition (LYZ 2006)

For $f \in W^{1,1}(\mathbb{R}^n)$, the optimal Sobolev body, $\langle f \rangle$, of f is defined as the unique origin-symmetric convex body such that

$$\int_{\mathbb{S}^{n-1}} g(u)\, dS(\langle f \rangle, u) = \int_{\mathbb{R}^n} g(\nabla f(x))\, dx$$

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Theorem (LYZ 2006)

For $f \in W^{1,1}(\mathbb{R}^n)$, the infimum over all origin-symmetric convex bodies K of volume $V(K) = v_n$ over

$$\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\, dx$$

is attained if and only if K is a dilate of $\langle f \rangle$.

Affine Sobolev inequality

Theorem (Gaoyong Zhang: JDG 1999)

For $f \in W^{1,1}(\mathbb{R}^n)$,

$$\frac{1}{n}\int_{\mathbb{S}^{n-1}}\Big(\int_{\mathbb{R}^n}|u\cdot\nabla f(x)|\,dx\Big)^{-n}du \leqslant \Big(\frac{v_n}{2\,v_{n-1}}\Big)^n\,|f|_{\frac{n}{n-1}}^{-n}.$$

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- Extended to $\mathrm{BV}(\mathbb{R}^n)$ by Tuo Wang (AIM 2012)

Valuations on Sobolev Spaces

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Theorem (L. 2012)

An operator $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle\mathcal{K}_c^n, \#\rangle$ *is a continuous and affinely covariant valuation*

$$\iff$$

$\exists\, c \geqslant 0$ *such that*

$$\mathrm{z}(f) = c\,\langle f\rangle$$

for every $f \in W^{1,1}(\mathbb{R}^n)$.

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Characterization of the Projection Body Operator

Theorem (L.: JDG 2010)

An operator $\mathrm{Z}:\mathcal{K}_0^n \to \langle \mathcal{K}^n, + \rangle$ *is a continuous valuation such that*

$$\mathrm{Z}(\phi K) = |\det \phi|\, \phi^{-t}\, \mathrm{Z}\, K \quad \forall\, \phi \in \mathrm{GL}(n)$$

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- Schuster & Wannerer: TAMS 2012; Haberl: JEMS 2011+; Abardia & Bernig: AIM 2011; Parapatits: TAMS 2012+

Sketch of the Proof

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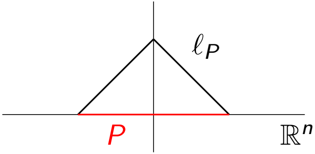
- $P^{1,1}(\mathbb{R}^n) \subset L^{1,1}(\mathbb{R}^n)$ 'linear elements'



$\ell_P \in L^{1,1}(\mathbb{R}^n)$

$P \in \mathcal{P}_0^n$

$\mathcal{P}_0^n$ convex polytopes in $\mathbb{R}^n$ containing the origin in their interiors

Sketch of the Proof, cont.

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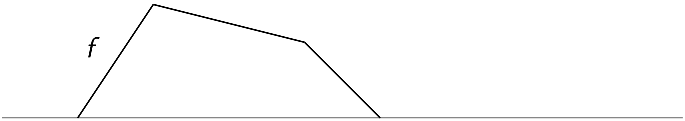

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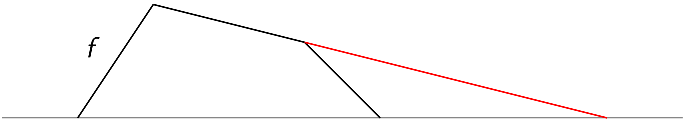

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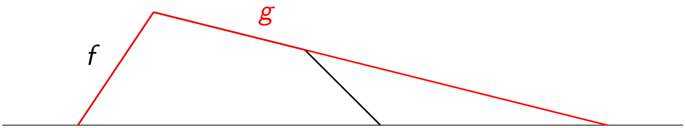

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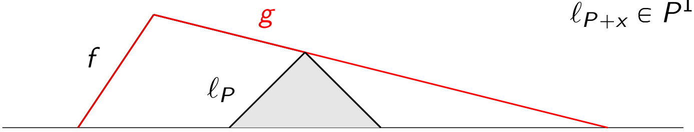

$\ell_{P+x} \in P^{1,1}(\mathbb{R}^n)$

Sketch of the Proof, cont.

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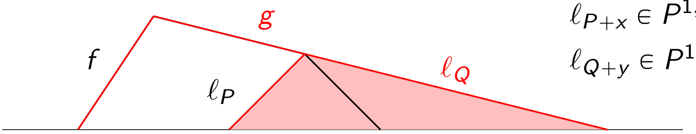

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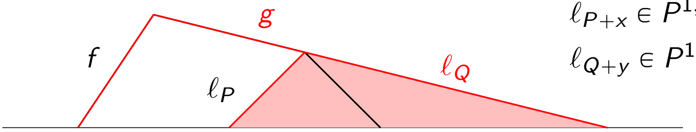

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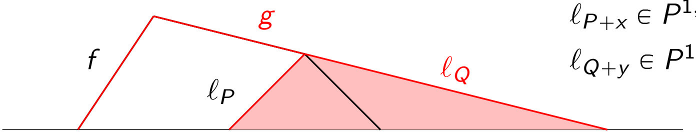

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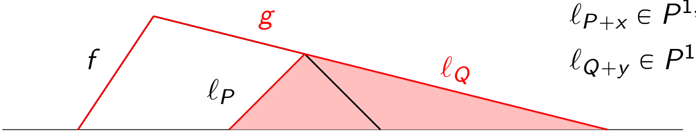

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- L.: JDG 2010 $\Rightarrow$ $\mathrm{z}(\ell_P) = c\, \Pi \langle \ell_P \rangle$

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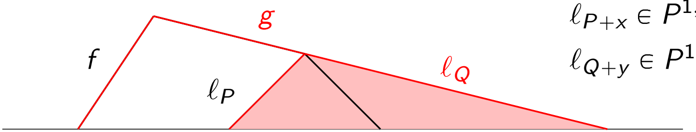

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- L.: JDG 2010 $\Rightarrow \mathrm{z}(\ell_P) = c\,\Pi\,\langle \ell_P \rangle$
- $\Rightarrow \mathrm{z}(f) = c\,\Pi\,\langle f \rangle$

Thank you !!!