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Universality for random matrices and log gases

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Universality for random matrices and log-gases

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Conference for András Krámli's 70-th birthday
University of Szeged, Jul 26-27, 2013

With P. Bourgade, B. Schlein, H.T. Yau, and J. Yin

Eugene Wigner (1956):

Successive energy levels of large nuclei have universal spacing statistics that can be described by random matrices. [Physics]

Freeman Dyson (1962):

A gas of particles with a logarithmic interaction reaches local equilibrium very fast. [Statistical Mechanics]

De Giorgi-Nash-Moser (1957-60):

Uniformly elliptic and parabolic equations in divergence form with rough coefficients have Hölder continuous solutions. [Math]

What do these facts have in common?

RANDOM MATRICES

Basic question [Wigner]: Is there some universal pattern in the eigenvalues of large random matrices?

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NN} \end{pmatrix} \Longrightarrow \quad (\lambda_1, \lambda_2, \dots, \lambda_N) \quad \text{eigenvalues?}$$

N = size of the matrix, will go to infinity.

Analogy: Central limit theorem: $\frac{1}{\sqrt{N}}(X_1 + X_2 + \ldots + X_N) \sim \mathcal{N}(0, \sigma^2)$

Gaussian Unitary Ensemble (GUE):

$H = (h_{jk})$ complex hermitian $N \times N$ matrix

$h_{jk} = \bar{h}_{kj}$ (for $j < k$) are complex and h_{kk} are real independent Gaussian random variables with normalization

$$\mathbb{E} h_{jk} = 0, \qquad \mathbb{E}|h_{jk}|^2 = \frac{1}{N}.$$

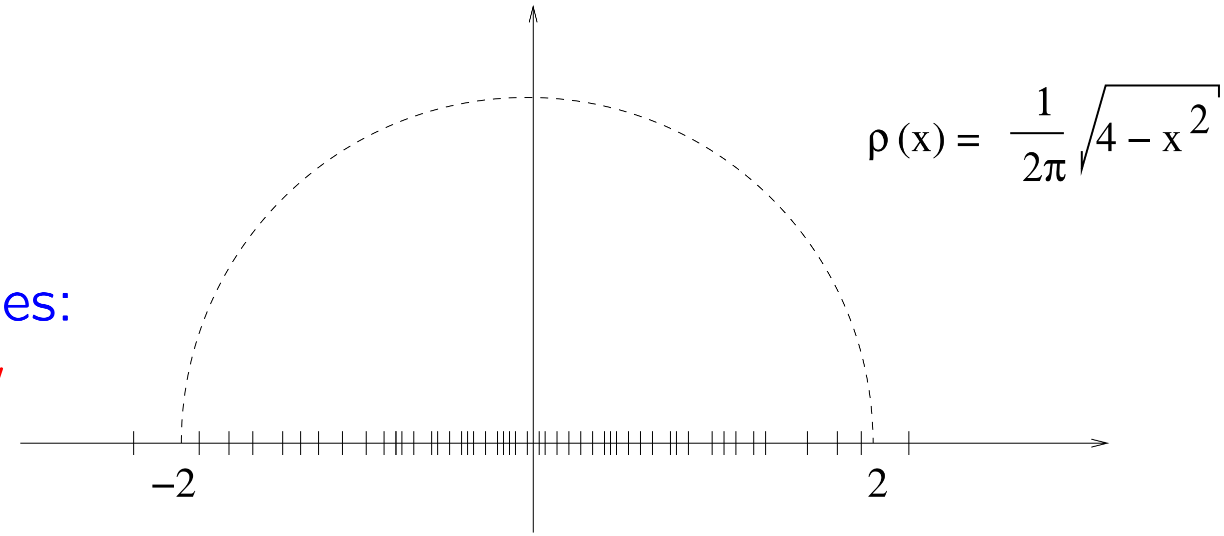
The eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ are of order one: (on average)

$$\mathbb{E}\frac{1}{N}\sum_i \lambda_i^2 = \mathbb{E}\frac{1}{N}\mathrm{Tr} H^2 = \frac{1}{N}\sum_{ij}\mathbb{E}|h_{ij}|^2 = 1$$

(Similar for GOE, GSE)

Wigner's observations

i) Density of eigenvalues:
Wigner semicircle law



ii) Level repulsion: Wigner surmise (in the bulk and for GOE)

$$\mathbb{P}\left(\lambda_{i+1} - \lambda_i = \frac{s}{N}\right) \approx \frac{\pi s}{2} \exp\left(-\frac{\pi}{4}s^2\right) \mathrm{d}s$$

Guessed by a 2x2 matrix calculation

SINE KERNEL FOR CORRELATION FUNCTIONS

Probability density of the eigenvalues: $p(x_1, x_2, \dots, x_N)$

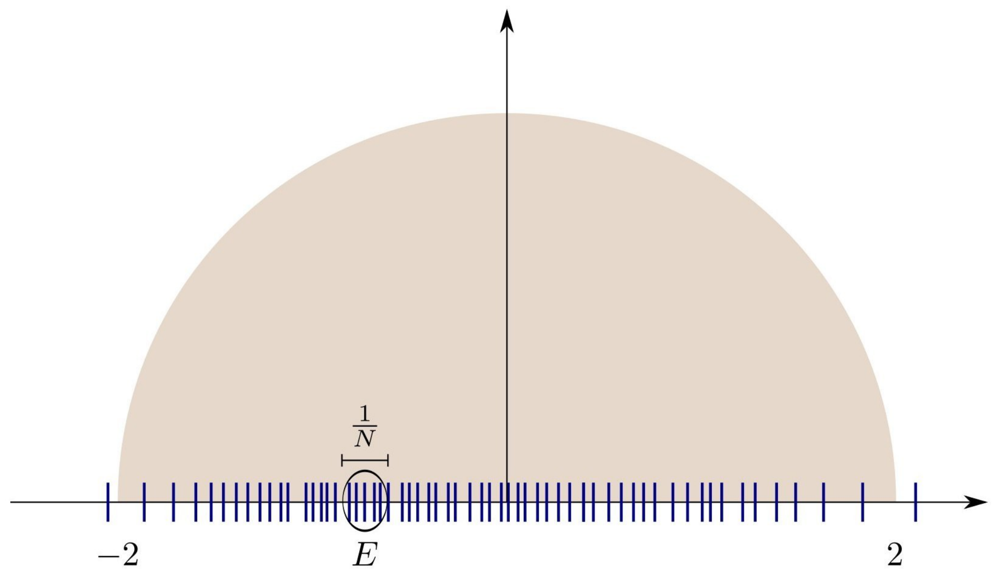
The k-point correlation function is given by

$$p_N^{(k)}(x_1, x_2, \dots, x_k) := \int_{\mathbb{R}^{N-k}} p(x_1, \dots x_k, x_{k+1}, \dots, x_N) \mathrm{d}x_{k+1} \dots \mathrm{d}x_N$$

Special case: $k = 1$ (density) is used to compute one-point observables

$$\mathbb{E} \frac{1}{N} \sum_{i=1}^{N} O(\lambda_i) = \int O(x) p_N^{(1)}(x) \mathrm{d}x \to \frac{1}{2\pi} \int O(x) \sqrt{4 - x^2} \mathrm{d}x$$

$p_N^{(k)}$ with higher k computes observables with k evalues.



Rescaled correlation functions at energy E

$$p_E^{(k)}(\mathbf{x}) := \frac{1}{[\varrho(E)]^k} p_N^{(k)}\Big(E + \frac{x_1}{N\varrho(E)}, E + \frac{x_2}{N\varrho(E)}, \ldots, E + \frac{x_k}{N\varrho(E)}\Big)$$

Rescales the gap $\lambda_{i+1} - \lambda_i$ to O(1).

Local level correlation statistics for GUE [Gaudin, Dyson, Mehta]

$$\lim_{N\to\infty} p_E^{(k)}(\mathbf{x}) = \det\left\{S(x_i - x_j)\right\}_{i,j=1}^{k}, \qquad S(x) := \frac{\sin \pi x}{\pi x}$$

$$p_E^{(2)}(\mathbf{x}) \to 1 - \left(\frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)}\right)^2 \qquad (\Longrightarrow \quad \text{Level repulsion})$$

The limit is independent of E as long as $|E| < 2$ (bulk spectrum)

Main question: going beyond Gaussian towards universality!

There are two disjoint directions of generalization: the Gaussian is the common intersection.

MODEL 1: INVARIANT ENSEMBLES (LOG-GAS)

Unitary ensemble: Hermitian matrices with density

$$\mathcal{P}(H)\mathrm{d}H \sim e^{-\mathrm{Tr}\, V(H)}\mathrm{d}H$$

Invariant under $H \to UHU^{-1}$ for any unitary U

Joint density function of the eigenvalues is explicitly known

$$p(\lambda_1, \ldots, \lambda_N) = \text{const.} \prod_{i<j} (\lambda_i - \lambda_j)^{\beta} e^{-\sum_j V(\lambda_j)}$$

classical ensembles $\beta = 1, 2, 4$ (orthogonal, unitary, symplectic symmetry classes; GOU, GUE, GSE for Gaussian case, $V(x) = x^2/2$)

General $\beta > 0$: Gibbs measure with inv. temp. β (no matrix):

$$\prod_{i<j} (\lambda_i - \lambda_j)^{\beta} e^{-\beta N \sum_i V(\lambda_i)} \sim e^{-\beta N \mathcal{H}(\lambda)}, \quad \mathcal{H} = \sum_i V(\lambda_i) - \frac{1}{N} \sum_{i<j} \log(\lambda_j - \lambda_i)$$

Prototype of strongly correlated stat.mech. system ("log-gas").

Universality conjecture for log-gases:

For any $\beta > 0$, the local statistics is independent of V

For classical $\beta = 1, 2, 4$, correlation fn's can be explicitly computed via large N asymptotics of orthogonal polynomials due to the Vandermonde determinant.

Dyson, Gaudin, Mehta: 60'-70's. Gaussian case (Hermite poly)

Deift, Pastur-Shcherbina, Bleher-Its, Lubinsky: from 90's, general case

For general β? New method is needed!

Main goal: develop new methods in statistical mechanics of strongly correlated systems.

MODEL 2: (GENERAL) WIGNER ENSEMBLES

$$H = (h_{ij})_{1 \le i,j \le N}, \quad \bar{h}_{ji} = h_{ij} \quad \text{independent}$$

$$\mathbb{E} h_{ij} = 0, \quad \mathbb{E}|h_{ij}|^2 = s_{ij}, \qquad \sum_i s_{ij} = 1,$$

$$\frac{c}{N} \le s_{ij} \le \frac{C}{N}$$

If h_{ij} are i.i.d. then it is called Wigner ensemble.

Universality conjecture (Dyson, Wigner, Mehta etc) : If h_{ij} are independent, then the local eigenvalue statistics are the same as for the Gaussian ensembles. Only symmetry type matters.

No previous results (apart from Johansson's for hermitian matrices with Gaussian convolution)

SOLUTION TO THE UNIVERSALITY CONJECTURES

Theorem [E-Schlein-Yau-Yin, 2009-2010] Local ev. statistics is universal for generalized Wigner ensembles in the bulk (and edge).

[Tao-Vu, 2010] Hermitian case via moment matching.

Theorem [Bourgade-E-Yau, 2011-2013] Let $\beta > 0$ and $V \in C^4$. Then local statistics is universal in the bulk.

Formally: weak convergence of the rescaled correlation functions

$$p_E^{(k)}(\mathbf{x}) \to \text{universal} \qquad\qquad \text{as } N \to \infty$$

Key ingredient: Uniqueness of the local Gibbs state with log-interaction

What else is left? Single gap – subtle difference from corr. fn's.

UNIVERSALITY OF GAPS

Theorem [E-Yau, 2012] Let $\beta \geq 1$ and V real analytic. Then single gap for the log-gas is universal in the bulk.

Precise formulation: Let μ_V and μ_G denote the β-log-gas with V and the Gaussian case. Then

$$\left| \mathbb{E}^{\mu_V} O\Big(N(\lambda_{i+1} - \lambda_i)\Big) - \mathbb{E}^{\mu_G} O\Big(N(\lambda_{j+1} - \lambda_j)\Big) \right| \leq CN^{-\varepsilon}$$

for any fixed label i, j in the bulk.

ε is a Hölder regularity exponent from De Giorgi-Nash-Moser theory!

Theorem [E-Yau, 2012] The single-gap universality holds for generalized Wigner ensembles in the bulk.

Previous result by Tao [2012] for GUE (explicit formulas) + hermitian ensembles with four moments matching GUE.

MAIN STEPS IN THE PROOF

i) Localization of the problem

ii) Representation as a random walk with long ranged jumps.

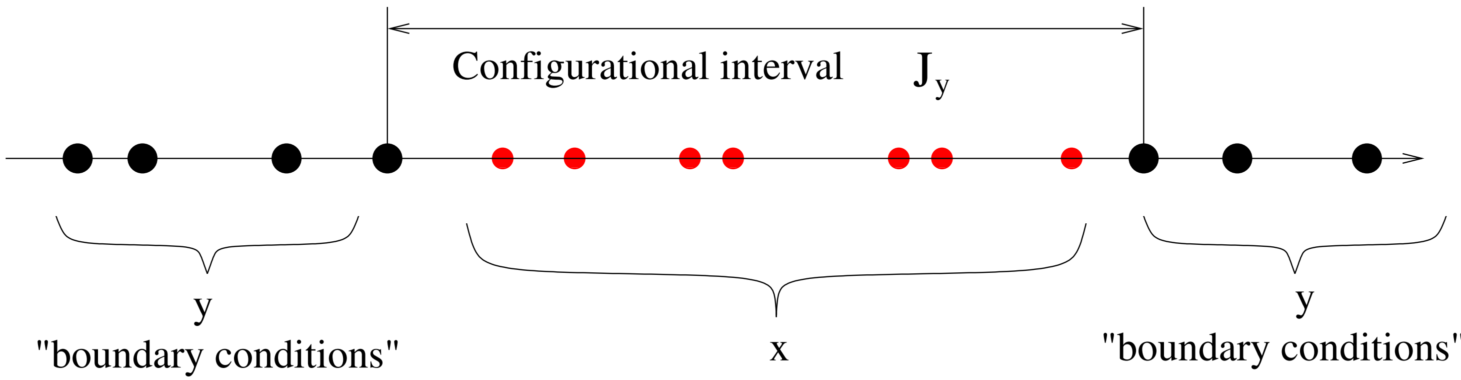
iii) Rigidity and level repulsion for the local equilibrium measure

iv) De Giorgi-Nash-Moser estimate with L^1 upper bound on the data

LOCAL MEASURE

Fix an interval of indices $I = [L-K, L+K]$ and rename the points

$$(\lambda_1, \lambda_2, \ldots, \lambda_N) := (y_1, \ldots y_{L-K-1}, x_{L-K}, \ldots, x_{L+K}, y_{L+K+1}, \ldots y_N)$$



Given $\mathbf{y}$ fixed, define the conditional measure $\mu_{\mathbf{y}}(\mathrm{d}\mathbf{x})$. It is again a β-log-gas, but with non-smooth potential and in $J_{\mathbf{y}}$.

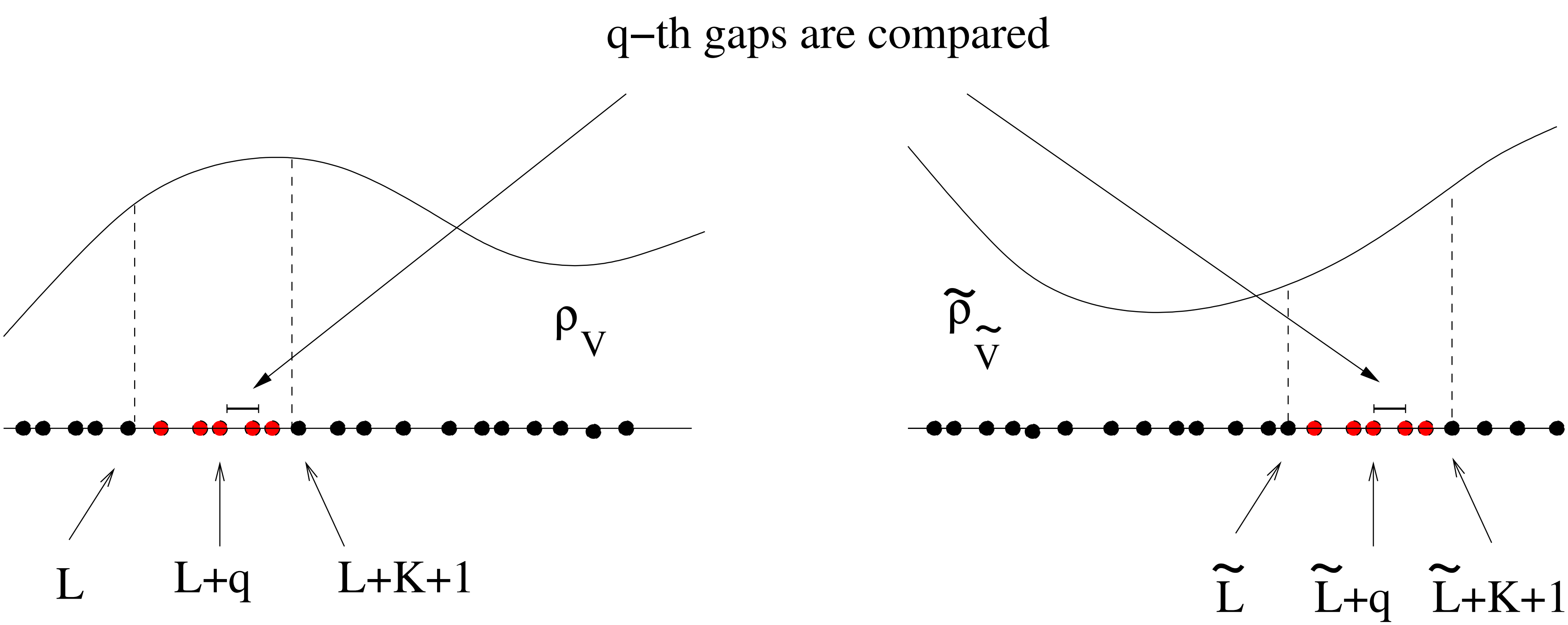
$$\mu_{\mathbf{y}} \sim e^{-N\beta\mathcal{H}_{\mathbf{y}}}, \qquad \mathcal{H}_{\mathbf{y}}(\mathbf{x}) := \sum_{i\in I} \frac{1}{2} V_{\mathbf{y}}(x_i) - \frac{1}{N} \sum_{i<j\in I} \log|x_j - x_i|$$

$$V_{\mathbf{y}}(x) := V(x) - \frac{2}{N} \sum_{k\notin I} \log|x - y_k| \qquad \text{(external potential)}$$

GAP UNIVERSALITY FOR THE LOCAL MEASURE

Take two β-log-gases with different potentials and localize them at different intervals. Assume the boundary conditions are "regular"

Theorem: [Informally] The local gaps have the same distribution.



INTERPOLATION BETWEEN μ AND $\widetilde{\mu}$

Given $\mu = \mu_{\mathbf{y},V}$ and $\widetilde{\mu} = \mu_{\widetilde{\mathbf{y}},\widetilde{V}}$, interpolate between them by defining

$$\omega^r_{\mathbf{y},\widetilde{\mathbf{y}}} := e^{-\beta r h(\mathbf{x})}\mu_{\mathbf{y}}, \qquad h(\mathbf{x}) := N\Big(\widetilde{V}_{\widetilde{\mathbf{y}}}(\mathbf{x}) - V_{\mathbf{y}}(\mathbf{x})\Big), \qquad r \in [0,1]$$

$$\Longrightarrow \qquad \Big|\Big[\mathbb{E}^{\mu} - \mathbb{E}^{\widetilde{\mu}}\Big] O(x_q - x_{q+1})\Big| = \beta \int_0^1 \mathrm{d}r \, \langle h; O(\ldots)\rangle_{\omega^r}$$

$\langle f; g\rangle = \mathbb{E}^{\omega} fg - \Big(\mathbb{E}^{\omega} f\Big)\Big(\mathbb{E}^{\omega} g\Big)$ is the covariance.

Need to show that

$$\langle h(\mathbf{x}); O(x_q - x_{q+1})\rangle_{\omega} \leq K^{-\varepsilon}. \qquad (1)$$

This is a statement about the decay of the “point-gap” covariance.

COVARIANCE STRUCTURE OF LOG-GASES

If μ is a log-gas on K points with interaction $\sum_{i<j} \log(x_j - x_i)$, then

$$\langle x_i ; x_j \rangle_\mu \sim \log \frac{K}{\left[|i-j| + 1 \right]} \qquad (2)$$

"point-point" covariance decays slowly. But "point-gap" is better:

$$\langle x_i \, ; \, x_j - x_{j+1} \rangle_\mu \sim \frac{\mathrm{d}}{\mathrm{d}j} \log \frac{K}{\left[|i-j| + 1 \right]} \sim \frac{1}{|i-j|} \qquad (3)$$

Only (2) is proved for GUE using explicit formulas [Gustaffson], the rest are conjectures!

Our main result (1) essentially proves (3) with $|i-j|^{-\varepsilon}$.

Usual methods for Gibbs measures do not apply due to the very strong correlation. Lack of methods!

RANDOM WALK REPRESENTATION

Let $\omega = e^{-\mathcal{H}}$ be any Gibbs measure and $\mathcal{L} = \Delta - \nabla\mathcal{H} \cdot \nabla$ its generator and $\mathbf{x}(t)$ is the stochastic process generated by $\mathcal{L}$, i.e.

$$\mathrm{d}\mathbf{x}(t) = \mathrm{d}\mathbf{B}(t) - \nabla\mathcal{H}(\mathbf{x}(t))\mathrm{d}t$$

Representation formula for the point-gap covariance:

$$\langle h(\mathbf{x}); O(x_q - x_{q+1})\rangle_\omega = \int_0^\infty \mathrm{d}\sigma \int \omega(\mathrm{d}\mathbf{x})\mathbb{E}_{\mathbf{x}}\Big[w_q(\sigma) - w_{q+1}(\sigma)\Big]O'(x_q - x_{q+1})$$

where $\mathbf{w}(s) = \mathbf{w}(s, \mathbf{x}(\cdot))$ (depending on the whole path) solves

$$\partial_s \mathbf{w}(s) = -\mathcal{A}(s)\mathbf{w}(s), \qquad \mathcal{A}(s) = \mathcal{H}''(\mathbf{x}(\sigma - s)), \qquad \mathbf{w}(0) = \nabla h(\mathbf{x}(\sigma)).$$

Similar formulas by [Helffer-Sjöstrand] and [Naddaf-Spencer] and ...
Note that $w_q - w_{q+1}$ is a discrete derivative!!

DE GIORGI-NASH-MOSER EMERGES

$$\langle h(\mathbf{x}); O(x_q - x_{q+1})\rangle_\omega = \int_0^\infty \mathrm{d}\sigma \int \omega(\mathrm{d}\mathbf{x})\mathbb{E}_{\mathbf{x}}\Big[w_q(\sigma) - w_{q+1}(\sigma)\Big] O'(x_q - x_{q+1})$$

$$\partial_s \mathbf{w}(s) = -\mathcal{A}(s)\mathbf{w}(s),$$

$$[\mathcal{A}(s)\mathbf{w}]_j = -\sum_i \mathcal{B}_{ij}(s)(w_i - w_j) + \text{potential}, \qquad \mathcal{B}_{ij}(s) = \frac{1}{(x_i(s) - x_j(s))^2}$$

If $x_i(s)$ were regularly spaced, then $\mathcal{B}_{ij}(s) \sim (i-j)^2$, i.e. the elliptic part $\mathcal{B}$ were the discretization of $|p| = \sqrt{-\Delta}$, and we had for the derivative of the fundamental solution

$$\int_0^\infty \mathrm{d}\sigma \Big[\Big(e^{-\sigma|p|}\Big)_{q,j} - \Big(e^{-\sigma|p|}\Big)_{q+1,j} \Big] = \nabla_q \Big(\frac{1}{|p|}\Big)_{j,q} \sim \frac{1}{|q-j|}$$

Adapting De Giorgi-Nash-Moser, in a recent paper Caffarelli *et. al.* proved Hölder regularity for a continuous version of this equation, where $\mathcal{B}_{ij}$ was replaced with $B(x-y) \sim |x-y|^{-2}$.

Lower bound (for ellipticity) on $\mathcal{B}_{ij} = \frac{1}{(x_i - x_j)^2}$ follows from $|x_i - x_j| \lesssim K^{\xi}|i-j|$ which is easy to guarantee by apriori location bound

Upper bound (needed in De Giorgi-Nash-Moser) is critical:

$$\mathcal{B}_{i,i+1}(s) \leq C \quad \Longleftrightarrow \quad |x_i(s) - x_{i+1}(s)| \geq C^{-1} \tag{4}$$

Level repulsion has only polynomial probability

$$\mathbb{P}^{\omega}(|x_i(s) - x_{i+1}(s)| \leq \varepsilon) \sim \varepsilon^{\beta}$$

so (4) cannot be guaranteed simultaneously for all s and i.

We develop De Giorgi-Nash-Moser with L^1 input on $\mathcal{B}_{ij}$.

SUMMARY

1. We proved bulk (and edge) universality for β-log-gases

2. We proved bulk (and edge) universality for generalized Wigner matrices (Wigner-Dyson-Mehta conjecture)

3. In both models we proved universality in both senses; averaged-energy and fixed gap.

4. We established the uniqueness of the Gibbs state for log-gases

OPEN QUESTIONS

1. Prove fixed energy universality beyond Hermitian case

2. Depart from mean field models via band matrices towards random Schrödinger

3. Understand general properties of log-gases, as a universal strongly correlated system.

BOLDOG SZÜLINAPOT ANDRÁS !!

Мой стих
        трудом
                громаду лет прорвет
и явится
        весомо,
                грубо,
                        зримо
как в наши дни
        вошел водопровод,
сработанный
        еще рабами Рима.

[М., В. В.]

$$\langle h(\mathbf{x}); O(x_q - x_{q+1})\rangle_\omega = \int_0^\infty \mathrm{d}\sigma \int \omega(\mathrm{d}\mathbf{x})\mathbb{E}_\mathbf{x}\Big[w_q(\sigma) - w_{q+1}(\sigma)\Big] O'(x_q - x_{q+1})$$

$$\partial_s \mathbf{w}(s) = -\mathcal{A}(s)\mathbf{w}(s), \qquad \mathcal{A}(s) = \mathcal{H}''(\mathbf{x}(\sigma - s)), \qquad w_j(0) = \nabla_j h$$

"Proof:" Using the commutator $[\mathcal{L}, \nabla] = \nabla \cdot \mathcal{H}''\nabla$, the covariance is

$$\langle f\,;\,g\rangle_\omega = (f, \mathcal{L}\frac{1}{\mathcal{L}}g)_\omega = (\nabla f, \nabla \frac{1}{\mathcal{L}} g)_\omega = \Big(\nabla f, \frac{1}{\mathcal{L} + \mathcal{H}''}\nabla g\Big)_\omega$$

Since $\mathbf{x}(t)$ is generated by $\mathcal{L}$, then by Feynman-Kac:

$$\langle f\,;\,g\rangle_\omega \asymp \int_0^\infty \mathrm{d}\sigma \Big(\nabla f(\mathbf{x}(\sigma)), e^{-\sigma\mathcal{H}''}\nabla g(\mathbf{x}(0))\Big)_\omega$$

Not quite true, since $\mathcal{H}'' = \mathcal{H}(\mathbf{x}(t))$ is time dependent.

Correctly: use the time dependent version of Feynman-Kac. $\square$