On the superintegrability of the rational Ruijsenaars-Schneider-van Diejen models[1]

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[1] Based on joint work with L. Fehér.

Content

Let us consider a Hamiltonian system (M, ω, h), where (M, ω) is a $2n$ dimensional symplectic manifold and h is the Hamiltonian.

Liouville integrability

The system is **Liouville integrable** if there exist n independent functions, $h_i \in C^\infty(M)$ $(i = 1, \dots, n)$; that mutually Poisson commute and the Hamiltonian h is equal to one of the h_i.

Maximal superintegrability

A Liouville integrable system is called **maximally superintegrable** if it has $(n-1)$ additional constants of motion (denote these as $f_j \in C^\infty(M)$) that are time independent, globally smooth and the $(2n-1)$ functions

$$h_1, \dots, h_n, f_1, \dots, f_{n-1}$$

are independent (their differentials are linearly independent) on a dense submanifold of M.

Examples of superintegrable systems

- Isotropic harmonic oscillator
- The Kepler problem and some magnetic analogues[a]
- Rational Calogero model[b]
- Rational Ruijsenaars-Schneider model[c]

[a] A. Ballesteros, A. Enciso, F.J. Herranz and O. Ragnisco, *Superintegrability on N-dimensional curved spaces: Central potentials, centrifugal terms and monopoles*, Ann. Phys. **324** (2009)

[b] S. Wojciechowski, *Superintegrability of the Calogero-Moser system*, Phys. Lett. **95A** (1983) 279-281

[c] V. Ayadi and L. Fehér, *On the superintegrability of the rational Ruijsenaars-Schneider model*, Phys.Let.A **374** (2010)

Remark: *Systems with repulsive interactions are expected to be superintegrable in general, but even for such systems it is interesting to exhibit extra constants of motion in explicit form.*

Superintegrability of Kepler problem

Let us consider the Hamiltonian system $(T^*\mathbb{R}^3, \sum_i dp_i \wedge dq_i, H)$, where the Hamiltonian is given by

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2m} p^2 - \frac{k}{q}\,.$$

The system is Liouville integrable: H, J_z, J^2 are the independent Poisson commuting constants of motions ($\mathbf{J} = \mathbf{q} \times \mathbf{p}$).
Additional constants of motions are the components of the Laplace-Runge-Lenz vector:

$$\mathbf{A} = \mathbf{p} \times \mathbf{J} - mk\hat{\mathbf{q}}.$$

Only two components of $\mathbf{A}$ are independent, since $\mathbf{A} \cdot \mathbf{J} = 0$.

With the constants of motions H, J_z, J^2, A_z, A^2 the Kepler problem is *maximally superintegrable*.

Let us focus on the n particle rational Calogero model defined by

$$H(x,p)=\frac{1}{2}\sum_{j=1}^{n}p_j^2+\sum_{j\neq k}\frac{\chi^2}{(x_j-x_k)^2}.$$

Following Wojciechowski[2], consider the functions

$$I_j=\mathrm{tr}(L^j)\qquad I_k^1=\mathrm{tr}(XL^{k+1})\,,\quad j,k\in\mathbb{Z}\,,j\geq 1,\ k\geq -1,$$

with the $n\times n$ matrices

$$X_{ij}=\delta_{ij}x_i\quad\text{and}\quad L_{ij}=\delta_{ij}p_i+(1-\delta_{ij})\,\mathrm{i}\,\frac{\chi}{(x_i-x_j)}\,.$$

[2] S. Wojciechowski, *Superintegrability of the Calogero-Moser system*, Phys. Lett. **95A** (1983) 279-281

The functions on the previous slide satisfy the Poisson bracket algebra

$$\{I_k, I_j\}_M = 0, \quad \{I_k^1, I_j\}_M = jI_{j+k}, \; \{I_k^1, I_j^1\}_M = (j-k)I_{k+j}^1 \, .$$

Wojciechowski used this algebra to show the superintegrability of the Calogero Hamiltonian given in terms of the Lax matrix as $H = I_2/2$.

Extra constants of motions

In fact, he found the following independent extra constants of motion

$$K_j = I_{j-2}^1 I_1 - I_{-1}^1 I_j \, , \quad j \in \{2, \dots, n\}.$$

We shall see later that the rational Calogero and RS models share the same "Wojciechowski algebra"

$$\{I_j, I_k\}_M = 0 \quad \{I_k^1, I_j\}_M = jI_{j+k} \quad \{I_k^1, I_j^1\}_M = (j-k)I_{k+j}^1 \, .$$

One difference is that in the RS case the functions I_j and I_k^1 are globally smooth **for all integers** j and k. In particular, in this case the functions I_k^1 realize the centerless Virasoro algebra over the full phase space.

Additional constants of motion

We can immediately see from the "Wojciechowski algebra" that

$$I_j^1\{I_k^1, I\}_M - I_k^1\{I_j^1, I\}_M$$

have vanishing Poisson brackets with I; for any $I = I(I_1, \dots, I_n)$.
In our cases the $2n$ functions

$$I_1, \dots, I_n, I_1^1, \dots, I_n^1$$

are independent. This means that the Jacobian

$$J = \det \frac{\partial(I_1, \dots, I_n, I_1^1, \dots, I_n^1)}{\partial(p_1, \dots, p_n, q_1, \dots, q_n)}$$

is non-vanishing generically.

Consequence: Superintegrability of I_j for all j=1, …, n

For any fixed j the functions

$$C_{k,j} := I_k^1 I_{2j} - I_j^1 I_{k+j}, \qquad k \in \{1,2,\dots,n\} \setminus \{j\}$$

Poisson commute with I_j. Using the functions $I_1, \dots I_n, I_1^1, \dots I_n^1$ as coordinates we can determine the Jacobian

$$J_j := \det \frac{\partial(I_a, C_{b,j})}{\partial(I_\alpha, I_\beta^1)}, \quad \text{where} \quad \begin{array}{l} a, \alpha \in \{1,\dots,n\} \\ b, \beta \in \{1,\dots,n\} \setminus \{j\} \end{array}.$$

The value of the determinant is $J_j = (I_{2j})^{n-1}$.
Since J_j is generically non-zero, it follows that I_j is superintegrable with the $(2n-1)$ independent functions

$$I_1, \dots, I_n\,, C_{k,j} \quad \text{, where} \quad k \in \{1, \dots n\} \setminus \{j\}.$$

The rational Ruijsenaars-Schneider model

The phase space of the model is

$$M = \mathcal{C}_n \times \mathbb{R}^n = \{(q,p) \,|\, q \in \mathcal{C}_n,\ p \in \mathbb{R}^n\},$$

where

$$\mathcal{C}_n := \{q \in \mathbb{R}^n \mid q_1 > q_2 > \cdots > q_n\}.$$

The symplectic structure $\omega = \sum_{k=1}^n dp_k \wedge dq_k$ corresponds to the fundamental Poisson brackets

$$\{q_i, p_j\}_M = \delta_{i,j}\,, \quad \{q_i, q_j\}_M = \{p_i, p_j\}_M = 0.$$

The rational RS Hamiltonian is given by

$$H_{RS} = \sum_{k=1}^n \cosh(p_k) \prod_{j \neq k} \left[1 + \frac{\chi^2}{(q_k - q_j)^2}\right]^{\frac{1}{2}}.$$

The rational RS model can be considered as a relativistic generalization of the rational Calogero model. To see this [3], introduce the functions:

$$P_{RS} = \sum_{k=1}^{n} \sinh(p_k) \prod_{j \neq k} \left[1 + \frac{\chi^2}{(q_k - q_j)^2}\right]^{\frac{1}{2}} ,$$

and

$$B_{RS} = -\sum_{i=1}^{n} q_i .$$

(H_{RS}, P_{RS}, B_{RS}) are the generators of the Poincaré algebra in $(1+1)$ dimension

$$\{H_{RS}, B_{RS}\}_M = P_{RS}, \quad \{P_{RS}, B_{RS}\}_M = H_{RS}, \quad \{H_{RS}, P_{RS}\}_M = 0.$$

[3] S.N.M. Ruijsenaars and H. Schneider, *A new class of integrable models and their relation to solitons*, Ann. Phys. **170** (1986)

The commuting Hamiltonians I_k are traces of the kth powers of the Lax matrix L. The Lax matrix (Hermitian, positive definite)

$$L(q,p)_{j,k} = u_j(q,p)\left[\frac{\mathrm{i}\,\chi}{\mathrm{i}\,\chi + (q_j - q_k)}\right] u_k(q,p)\,,$$

with the $\mathbb{R}_+$ valued functions

$$u_j(q,p) := e^{p_j}\prod_{m\neq j}\left[1+\frac{\chi^2}{(q_j-q_m)^2}\right]^{\frac{1}{4}}\,.$$

We define the functions $I_j, I_k^1 \in C^\infty(M)$ by

$$I_j(q,p) := \mathrm{tr}\left(L(q,p)^j\right), \quad I_k^1(q,p) := \mathrm{tr}\left(L(q,p)^k \mathbf{q}\right) \quad \forall j,k\in\mathbb{Z}\,,$$

using the diagonal matrix $\mathbf{q} := \mathrm{diag}(q_1,\ldots,q_n)$.

The functions I_j, I_k^1 satisfy the same "Wojciechowski Poisson bracket relations" that were mentioned previously.

The RS Hamiltonian's expression in terms of I_k-s

$$H_{RS} = (1/2)(I_1 + I_{-1}).$$

Extra constants of motion

Examining the expression $I_j^1\{I_1^1, H_{RS}\}_M - I_1^1\{I_j^1, H_{RS}\}_M$, we get extra constants of motion for the superintegrability of H_{RS}. These are given by

$$K_j := I_j^1(I_2 - n) - I_1^1(I_{j+1} - I_{j-1}), \quad j = 2, \ldots, n.$$

We can show that $I_1, \ldots, I_n, I_1^1, \ldots, I_n^1$ generically form coordinates on the phase space, so they are functionally independent.

We immediately obtain the following equation

$$\det \frac{\partial(I_a, K_b)}{\partial(I_\alpha, I_\beta^1)} = (I_2 - n)^{n-1}, \qquad \begin{array}{l} a, \alpha \in \{1, \ldots, n\} \\ b, \beta \in \{2, \ldots, n\} \end{array},$$

which is generically non-vanishing. This implies that the $(2n-1)$ function I_a , K_b are independent.

Thus we've proved the superintegrability of the Hamiltonian H_{RS}.

We've established the "Wojciechowski Poisson algebra" using the derivation of rational RS model by Hamiltonian reduction[a].
Next we describe the basic idea of this procedure.

[a] L. Fehér and C. Klimčík, *On the duality between the hyperbolic Sutherland and the rational Ruijsenaars-Schneider models*, J. Phys. A: Math. Theor. **42** (2009)

The unreduced phase space

$$T^*GL(n,\mathbb{C}) \times \mathcal{O}_\chi \equiv GL(n,\mathbb{C}) \times gl(n,\mathbb{C}) \times \mathcal{O}_\chi = \{(g, J^R, \xi)\}\,.$$

Here $\mathcal{O}_\chi$ is a minimal coadjoint orbit of the group $U(n)$

$$\mathcal{O}_\chi := \{\mathrm{i}\,\chi(\mathbf{1}_n - vv^\dagger) \mid v \in \mathbb{C}^n,\ |v|^2 = n\}.$$

The symplectic form is

$$\Omega = d\langle J^R, g^{-1}dg\rangle + \omega^{\mathcal{O}_\chi}\,.$$

The corresponding Poisson-brackets are

$$\{g, \langle X, J^R\rangle\} = gX, \quad \{\langle X, J^R\rangle, \langle Y, J^R\rangle\} = -\langle [X,Y], J^R\rangle,$$
$$\{\langle X_+, \xi\rangle, \langle Y_+, \xi\rangle\} = \langle [X_+, Y_+], \xi\rangle,$$

where X_+, Y_+ are the anti-hermitian parts of matrices $X, Y \in gl(n,\mathbb{C})$.

Symmetry group

We reduce using the symmetry group $K := U(n) \times U(n)$; for an element $(\eta_L, \eta_R) \in K$ the action is given by

$$\Psi_{(\eta_L,\eta_R)} : (g, J^R, \xi) \mapsto (\eta_L g \eta_R^{-1}, \eta_R J^R \eta_R^{-1}, \eta_L \xi \eta_L^{-1}).$$

Momentum constraint

We set the momentum map corresponding to the action to zero, in other words we introduce first class constraints

$$J_+^R = 0 \qquad \text{and} \qquad (g J^R g^{-1})_+ + \xi = 0.$$

Reduced phase space

The resulting reduced phase space can be identified with the Ruijsenaars-Schneider phase space (M, ω). In fact the following manifold is a global gauge-slice

$$S := \{(L(q,p)^{\frac{1}{2}}, -2\mathbf{q}, \xi(q,p)) \,|\, (q,p) \in \mathcal{C}_n \times \mathbb{R}^n\}\,.$$

Here ξ is an $\mathcal{O}_\chi$ valued function on M:

$$\xi(q,p) := \mathrm{i}\,\chi(\mathbf{1}_n - v(q,p)v(q,p)^\dagger),$$

where $v(q,p) := L(q,p)^{-\frac{1}{2}} u(q,p)$, with the Lax matrix L.

We can realize the functions $I_j, I_k^1 \in C^\infty(M)$ as restrictions of K-invariant functions $\mathcal{I}_j, \mathcal{I}_k^1$ to S. These functions are given by

$$\mathcal{I}_j(g, J^R, \xi) := \operatorname{tr}\big((g^\dagger g)^j\big),\ \mathcal{I}_k^1(g, J^R, \xi) := -\frac{1}{2}\Re \operatorname{tr}\big((g^\dagger g)^k J^R\big)\,.$$

The K-invariant functions survive the reduction. We can easily verify the relations

$$\{\mathcal{I}_j, \mathcal{I}_k\} = 0\,, \quad \{\mathcal{I}_k^1, \mathcal{I}_j\} = j\,\mathcal{I}_{j+k}\,, \quad \{\mathcal{I}_k^1, \mathcal{I}_j^1\} = (j-k)\,\mathcal{I}_{k+j}^1$$

for all $j, k \in \mathbb{Z}$. These relations imply

$$\{I_j^1, I_k\}_M = 0\,, \quad \{I_k^1, I_j\}_M = jI_{j+k}\,, \quad \{I_k^1, I_j^1\}_M = (j-k)I_{k+j}^1\,.$$

The rational Ruijsenaars-Schneider-van Diejen model

The $BC(n)$ generalization of the rational RS model is due to van Diejen.[a]
The Hamiltonian is given by

$$H_{RSvD} = \sum_{c=1}^{n} \cosh(2\theta_c) \left(1 + \frac{\nu^2}{\lambda_c^2}\right)^{\frac{1}{2}} \left(1 + \frac{\kappa^2}{\lambda_c^2}\right)^{\frac{1}{2}} \prod_{\substack{d=1 \\ (d \neq c)}}^{n} \left(1 + \frac{4\mu^2}{(\lambda_c - \lambda_d)^2}\right)^{\frac{1}{2}} \left(1 + \frac{4\mu^2}{(\lambda_c + \lambda_d)^2}\right)^{\frac{1}{2}} + \frac{\nu\kappa}{4\mu^2} \prod_{c=1}^{n} \left(1 + \frac{4\mu^2}{\lambda_c^2}\right) - \frac{\nu\kappa}{4\mu^2},$$

with κ, μ, ν real parameters that satisfy $\mu \neq 0 \neq \nu$ and $\kappa\nu \geq 0$.

[a] J.F. van Diejen, Deformations of Calogero–Moser systems and finite Toda chains, Theor. Math. Phys. **99** (1994)

The RSvD phase space

The phase space of the rational RSvD model is given by

$$P = \mathcal{C}_n \times \mathbb{R}^n = \{(\lambda, \theta) \,|\, \lambda \in \mathcal{C}_n, \, \theta \in \mathbb{R}^n\},$$

where we've introduced the notation

$$\mathcal{C}_n := \{\lambda \in \mathbb{R}^n \,|\, \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0\}.$$

The symplectic structure $\omega = -2\sum_{k=1}^{n} d\theta_k \wedge d\lambda_k$ corresponds to the fundamental Poisson brackets

$$2\{\theta_i, \lambda_j\}_P = \delta_{i,j}, \quad \{\lambda_i, \lambda_j\}_P = \{\theta_i, \theta_j\}_P = 0.$$

First we need to introduce some matrix valued functions on the phase space on the next few slides. We'll rely on Pusztai's article[a]. He derived the RSvD model by Hamiltonian reduction.

[a]B. G. Pusztai, *The hyperbolic BC(n) Sutherland and the rational BC(n) Ruijsenaars-Schneider-van Diejen models: Lax matrices and duality*, Nucl. Phys. B **856** (2012)

Then we'll explicitly construct the "Wojciechowski algebra" and the extra constants of motions.

$\mathcal{A}(\lambda, \theta)$ is a $2n \times 2n$ Hermitian matrix, where the matrix entries lying in the diagonal $n \times n$ blocks are given by the formulae

$$\mathcal{A}_{a,b}(\lambda, \theta) = e^{\theta_a+\theta_b}|z_a(\lambda)z_b(\lambda)|^{\frac{1}{2}} \frac{2\,\mathrm{i}\,\mu}{2\,\mathrm{i}\,\mu + \lambda_a - \lambda_b},$$

$$\mathcal{A}_{n+a,n+b}(\lambda, \theta) = e^{-\theta_a-\theta_b} \frac{\overline{z_a(\lambda)}z_b(\lambda)}{|z_a(\lambda)z_b(\lambda)|^{\frac{1}{2}}} \frac{2\,\mathrm{i}\,\mu}{2\,\mathrm{i}\,\mu - \lambda_a + \lambda_b},$$

and the matrix entries of the $n \times n$ off-diagonal blocks have the form

$$\mathcal{A}_{a,n+b}(\lambda, \theta) = \overline{\mathcal{A}_{n+b,a}(\lambda, \theta)}$$
$$= e^{\theta_a-\theta_b} z_b(\lambda)|z_a(\lambda)z_b(\lambda)^{-1}|^{\frac{1}{2}} \frac{2\,\mathrm{i}\,\mu}{2\,\mathrm{i}\,\mu + \lambda_a + \lambda_b} + \frac{\mathrm{i}(\mu - \nu)}{\mathrm{i}\,\mu + \lambda_a}\delta_{a,b},$$

for any $a, b \in \{1, \dots, n\}$. Here we use the functions

$$\mathcal{C}_n \ni \lambda \to z_a(\lambda) = -\left(1 + \frac{\mathrm{i}\,\nu}{\lambda_a}\right) \prod_{\substack{d=1\\ d\neq a}}^{n} \left(1 + \frac{2\,\mathrm{i}\,\mu}{\lambda_a - \lambda_d}\right)\left(1 + \frac{2\,\mathrm{i}\,\mu}{\lambda_a + \lambda_d}\right).$$

Let us define the $n \times n$ diagonal matrix $\boldsymbol{\lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Now the Hermitian $2n \times 2n$ matrix $h(\boldsymbol{\lambda})$ is provided by

$$h(\lambda) = \begin{bmatrix} \alpha(\boldsymbol{\lambda}) & \beta(\boldsymbol{\lambda}) \\ -\beta(\boldsymbol{\lambda}) & \alpha(\boldsymbol{\lambda}) \end{bmatrix},$$

with the functions

$$\alpha(x) = \frac{\sqrt{x + \sqrt{x^2 + \kappa^2}}}{\sqrt{2x}} \quad \text{and} \quad \beta(x) = \mathrm{i}\,\kappa \frac{1}{\sqrt{2x}} \frac{1}{\sqrt{x + \sqrt{x^2 + \kappa^2}}},$$

where $x \in (0, \infty)$. Introduce also the diagonal matrix

$$\Lambda := \mathrm{diag}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots - \lambda_n).$$

Let us define the functions $I_j, I_k^1 \in C^\infty(P)$ by

$$I_j(\lambda, \theta) = \mathrm{tr}[(h^{-1}\,\mathcal{A}\,h^{-1})^j]$$
$$I_k^1(\lambda, \theta) = (-1/4)\,\mathrm{tr}[(h\Lambda h^{-1} + h^{-1}\Lambda h)(h^{-1}\,\mathcal{A}\,h^{-1})^k]\,,$$

for all $j, k \in \mathbb{Z}$.

Using Pusztai's work, the Poisson bracket relations of the functions I_j , I_k^1 can be determined in a similar way as in the previous model.
In this case we obtain the same "Wojciechowski algebra" again.

The principal Hamiltonian can be expressed according to

$$H_{RSvD} = (1/2)I_1 \,.$$

Then $(n-1)$ independent extra constants of motions are given by

$$K_j = I_j^1 I_2 - I_1^1 I_{j+1} \,, \quad j \in \{2, \dots, n\}.$$

Thus we've proved the superintegrability of the RSvD model.

Result

We've shown explicitly the superintegrability of the rational A_n Ruijsenaars-Schneider model and the rational BC_n RSvD model

Remark

The functions I_k^1 have linear time dependence under the Hamiltonian flow of $I = I(I_1, \ldots, I_n)$. Thus we have obtained an explicit linearisation of the dynamics.

Thank you for your attention!

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