# Binary Tomography Using Two Projections and Morphological Skeleton

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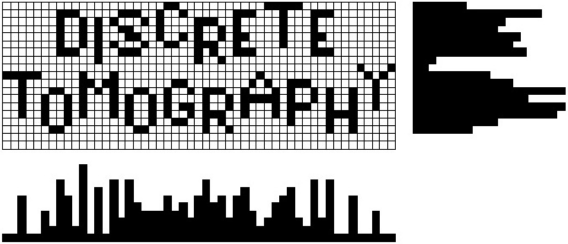
CS$^2$

# Contents

# Motivation

Discrete tomography reconstruct (discrete) images of objects from their projections

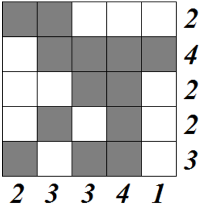



Extremely ambiguous if only a few projections are available
→ further information is needed

# Image and projections

**Image** binary square matrix, $F_{n \times n}$

**Projections** sum of rows and columns, $\mathcal{H}(F)$, $\mathcal{V}(F)$



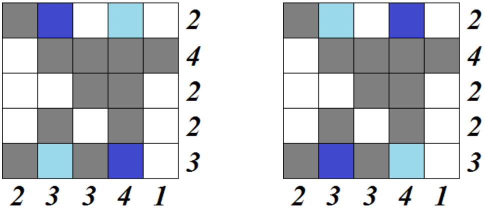

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\mathcal{H}_i(F) = \sum_{j=1}^{n} F_{ij} \qquad \mathcal{V}_j(F) = \sum_{i=1}^{n} F_{ij}$$

# Switching components

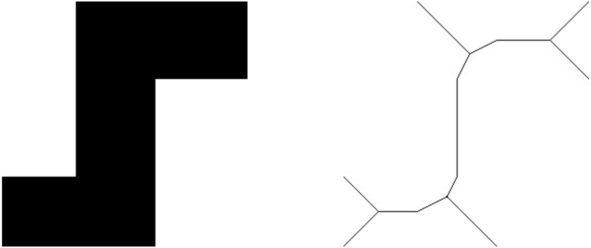
- Submatrix of an image in size of $2 \times 2$ where switching 0-s and 1-s do not change the projections
- Necessary and sufficent condition for ambiguity

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



# Morphological skeleton

- Representation of shapes
- Close to the topological skeleton
- Easy to generate
- Can store the image uniquely (with some additional information)



# Morphological skeleton

The morphological skeleton of the binary image $F$ with the structuring element $Y$ can be extracted with morphological operators (erosion and dilation).

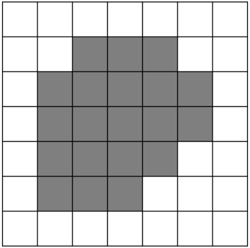
$$S(F,Y) = \bigcup_k S_k(F,Y),$$

where

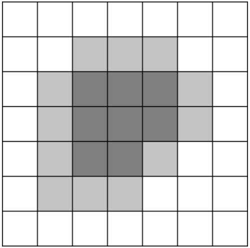
$$S_k(F,Y) = (F \ominus_k Y) - \left[(F \ominus_{k+1} Y) \oplus Y\right].$$

# Morphological skeleton

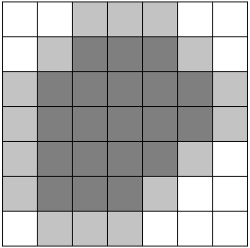
## Example for erosion and dilation



Original $F$



Erosion $F \ominus Y$



Dilation $F \oplus Y$

# Morphological skeleton

If we know the structuring element $Y$ and the $S_k(F, Y)$ for each $k$, then we can reconstruct the orignal image:

$$F = \bigcup_k \left[ S_k(F, Y) \oplus_k Y \right],$$

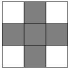
or in a different form:

$$F = \bigcup_{p \in S(F,Y)} (p \oplus_{k_p} Y),$$

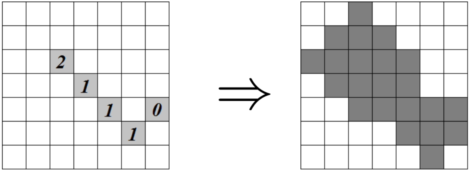
where $k_p$ is a unique value for every $p$ such that $p \in S_{k_p}(F)$.

# Morphological skeleton

Let $K(S) := (k_{p_1}, k_{p_2}, ..., k_{p_{|S|}})$ the series of the $k_p$ values, $p_i \in S = S(F, Y)$, and let the structuring element $Y := \{(-1, 0), (0, -1), (0, 0), (0, 1), (1, 0)\}$.



$F$ is uniquely determined by $K(S)$ and $Y$.

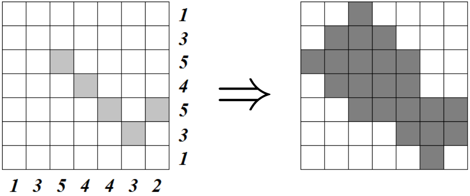


# Skeleton based reconstruction

## Main task

Given projections $H$ and $V$, morphological skeleton $S$ and structuring element $Y$. We want to find $K(S)$ in a way that the corresponding $F$ is the closest to the required projections:

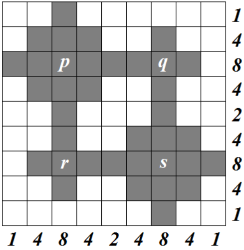
$$f(K(S)) = ||H - \mathcal{H}(F)||_2 + ||V - \mathcal{V}(F)||_2 \rightarrow \min$$

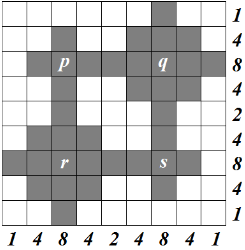


# Theoretical results

**Theorem (Un-uniqueness)**

The skeleton based reconstruction is not unique, since there could be an image pair $F_1$ and $F_2$ that they have the same projections and skeleton, however $F_1 \neq F_2$.




Note that $(2,1,1,2) = K_1(S) \quad \neq \quad K_2(S) = (1,2,2,1)$.

# Theoretical results

## Theorem (Skeletal smoothness)

For any image $F$ and any skeletal points $p, q \in S(F, Y)$,

$$|k_p - k_q| < ||p - q||_1,$$

where $||.||_1$ denotes the Manhattan norm.

As a special case, if $p$ is 8-adjacent to $q$, then $|k_p - k_q| < 2$, that we are to use during the reconstruction.

# Theoretical results

Given $H, V$ projection vectors, $S$ skeletal set. Does any binary image $F$ exist where $H = \mathcal{H}(F)$, $V = \mathcal{V}(F)$, $S = S(F, Y)$ and $F$ is 4-connected?

**Theorem (NP-completedness)**

The problem above is NP-complete.

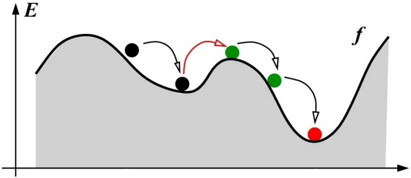
Note that we fixed the structuring element $Y$.

**Conjecture**

The problem above is still NP-complete even without requiring the 4-connectedness.

# Simulated Annealing

- Iterative stochastic method for finding a global minimum of a function
- Could find a near-optimal minimum in a reasonable time
- Has many technical parameters, in our case:
  - Variables: $K(S)$
  - Energy function: $f\big(K(S)\big) \to \min$
  - Stopping criteria: iteration number $M$ or zero energy
  - Annealing schedule: $T(t) = T_0 \cdot \left(\frac{T_s}{T_0}\right)^{\frac{t}{M}}$, where $t$ denotes time



# Simulated Annealing
## NVC

NVC *(No Vase Constraint)* model:

- $f\big(K(S)\big) = ||H - \mathcal{H}(F)||_2 + ||V - \mathcal{V}(F)||_2$
- Changing a variable: simply change an element $k_p \in K(S)$ randomly

# Simulated Annealing
## DVC

DVC *(Dynamic Vase Constraint)* model:

- $f$ is the same as in NVC
- Changing a variable: change an element $k_p \in K(S)$ such that $|k_p - k_q| < C(t)$ for each $q$ 8-adjacent to $p$ and

$$C(t) = \left\lceil C_0 \cdot \left(\frac{C_s}{C_0}\right)^{\frac{t}{M}} \right\rceil.$$

Note that $C(t)$ is monotonically decreasing and the limit is 1

# Simulated Annealing
## CEF

CEF *(Combined Energy Function)* model:

- $$f\big(K(S)\big) = \alpha \cdot \Big( ||H - \mathcal{H}(F)||_2 + ||V - \mathcal{V}(F)||_2 \Big) + \\ + (1-\alpha) \cdot \sum_{||p-q||_1 \le 1} h(k_p, k_q) \ ,$$

  where

  $$h(k_p, k_q) = \begin{cases} 0 & \text{if } |k_p - k_q| \le 1 \\ |k_p - k_q|/2 & \text{otherwise.} \end{cases}$$

- Changing a variable: the same as in NVC

# Results

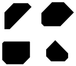
- Artificial images in size of $256 \times 256$
  1. Simple convex shape
  2. Grid of comvex shapes
  3. Random set of convex shapes
  4. Miscellaneous images
- Technical parameters: $M = 50000$, $T_0 = 10$, $T_s = 0.001$
- Testing environment: Intel Core 2 Duo T250, 1.5 GHz, 2GB RAM

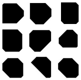
## Error measurement

$$E = \sqrt{\sum_{i=1}^{2n} (b_i - b'_i)^2} \ ,$$

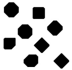
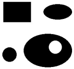
where $b_i$ and $b'_i$ are the elements of the original and the reconstructed projections, respectively.

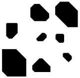
# Results

| Image | Method | CPU | $E$ |
|---|---|---|---|
|  | NVC | 3842 | 1060 |
| | $DVC_{10}$ | 4030 | 98 |
| | $DVC_5$ | 4116 | 97 |
| | **$DVC_1$** | **4563** | **18** |
| | $CEF_{0.3}$ | 4358 | 2468 |
| | $CEF_{0.5}$ | 4415 | 1675 |
| | $CEF_{0.7}$ | 4435 | 1305 |
|  | NVC | 3784 | 3405 |
| | **$DVC_{10}$** | **3038** | **1291** |
| | $DVC_5$ | 3164 | 4288 |
| | $DVC_1$ | 3566 | 5307 |
| | $CEF_{0.3}$ | 5412 | 5665 |
| | $CEF_{0.5}$ | 5387 | 4829 |
| | $CEF_{0.7}$ | 5328 | 3212 |

| Image | Method | CPU | $E$ |
|---|---|---|---|
|  | NVC | 7276 | 1285 |
| | $DVC_{10}$ | 7900 | 174 |
| | $DVC_5$ | 8127 | 146 |
| | **$DVC_1$** | **4473** | **0** |
| | $CEF_{0.3}$ | 7626 | 2578 |
| | $CEF_{0.5}$ | 7665 | 1849 |
| | $CEF_{0.7}$ | 7691 | 1505 |
|  | **NVC** | **4346** | **6136** |
| | $DVC_{10}$ | 4733 | 1066145 |
| | $DVC_5$ | 4609 | 1722350 |
| | $DVC_1$ | 4926 | 3302481 |
| | $CEF_{0.3}$ | 7308 | 14371 |
| | $CEF_{0.5}$ | 7243 | 8896 |
| | $CEF_{0.7}$ | 7222 | 7402 |

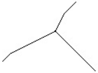
# Results

| Image | Method | CPU | $E$ |
|---|---|---|---|
|  | NVC | 1666 | 1341 |
| | **$DVC_{10}$** | **1215** | **292** |
| | $DVC_5$ | 1234 | 314 |
| | $DVC_1$ | 1302 | 294 |
| | $CEF_{0.3}$ | 2904 | 2534 |
| | $CEF_{0.5}$ | 2827 | 1950 |
| | $CEF_{0.7}$ | 2851 | 1732 |
|  | **NVC** | **3537** | **2530** |
| | $DVC_{10}$ | 2852 | 9154 |
| | $DVC_5$ | 2981 | 13138 |
| | $DVC_1$ | 3226 | 67493 |
| | $CEF_{0.3}$ | 6380 | 5183 |
| | $CEF_{0.5}$ | 6367 | 4102 |
| | $CEF_{0.7}$ | 6343 | 3029 |

| Image | Method | CPU | $E$ |
|---|---|---|---|
|  | **NVC** | **2165** | **2709** |
| | $DVC_{10}$ | 1713 | 6042 |
| | $DVC_5$ | 1724 | 7962 |
| | $DVC_1$ | 1910 | 6360 |
| | $CEF_{0.3}$ | 4123 | 5688 |
| | $CEF_{0.5}$ | 4131 | 4178 |
| | $CEF_{0.7}$ | 4114 | 3346 |
|  | **NVC** | **2757** | **4034** |
| | $DVC_{10}$ | 2304 | 4523 |
| | $DVC_5$ | 2467 | 7472 |
| | $DVC_1$ | 2430 | 13096 |
| | $CEF_{0.3}$ | 8884 | 6663 |
| | $CEF_{0.5}$ | 8856 | 5012 |
| | $CEF_{0.7}$ | 8959 | 4407 |

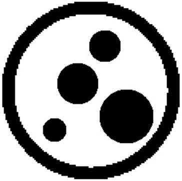
# Results
## Examples

Original

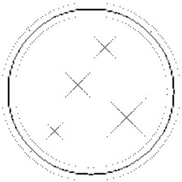


Skeleton

Result with $\mathrm{CEF}_{0.5}$



Original



Skeleton

Result with NVC

# Conclusions

- Image reconstruction is extremely underdetermined if only a few projections are used
- Morphological skeleton can reduce the ambiguity, however, the reconstruction problem is (possibly) NP-complete
- 3 variants of SA are tested on artifical images
  - NVC generally acceptable reconstruction
  - DVC smoother results, sometimes converges very slowly (highly depends on the initial image)
  - CEF similar results as NVC, computationally intensive

# Future work

- Prove NP- (or P-) completedness of the original task and its variants (such as $h$-convex images)
- Examine strategies for choosing the initial image for SA
- Find a more sophisticated function minimizer
- Try other prior information, such as smoothness on the boundary
- Study the robustness of the reconstruction when the projections are corrupted by noise

# References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HUNGARY'S RENEWAL

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