

# Quantum theory of light-matter interaction: Fundamentals

## Lecture 8

### The two-level coherent resonant approximation, the optical Rabi problem

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# Introduction

In this Lecture we consider again the dynamics of an atom under the influence of a given external, linearly polarized harmonic electric field of circular frequency  $\omega$  and amplitude  $E_0$ :

$$E(t) = E_0 \cos \omega t.$$

We have seen that perturbation theory fails if the system goes away significantly from the initial state, i.e. when  $C_f$  gets close to unity. We know from perturbation theory, or already from the elements (Bohr's theory) that this happens if the field induces transition between *two resonant energy levels*, which means that there are two atomic stationary states with energies  $\varepsilon_1 < \varepsilon_2$ , such that their difference  $\varepsilon_2 - \varepsilon_1 = \hbar\omega_{21} =: \hbar\omega_0$  is close to  $\hbar\omega$ . This condition can be expressed as:

$$|\omega_0 - \omega| =: |\Delta| \ll \omega_0.$$

## Two-level resonant approximation

Then to a very good approximation one can neglect all the other levels and the problem can be solved even if one allows that both levels involved may be populated nonnegligibly. We also assume here that these levels are nondegenerate, i. e. there is only one stationary state belonging to each energy to be denoted here by  $|u_1\rangle$  and  $|u_2\rangle$ . They can be for instance the  $1s$  and  $2p_z$  states of a hydrogen atom, assumed to be time independent. This means that we have

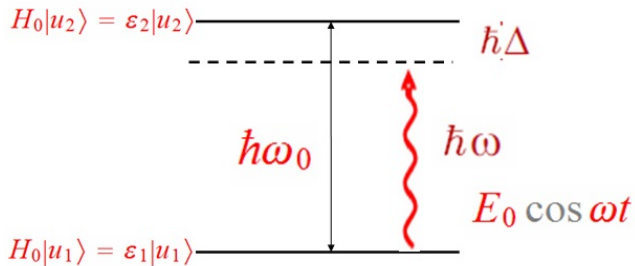
$$H_0 |u_1\rangle = \varepsilon_1 |u_1\rangle,$$

$$H_0 |u_2\rangle = \varepsilon_2 |u_2\rangle,$$

where  $H_0$  is the atomic Hamiltonian without the interaction with the field

The assumptions given above comprise the *two-level and resonant approximation*.

## Two resonant levels



## Schrödinger Equation for the C amplitudes

With the formalism of the previous lecture, we have now only two coefficients in  $|\Psi(t)\rangle$ :

$$|\Psi(t)\rangle = C_1(t)e^{-i\frac{\varepsilon_1}{\hbar}t} |u_1\rangle + C_2(t)e^{-i\frac{\varepsilon_2}{\hbar}t} |u_2\rangle .$$

The interaction Hamiltonian reads:

$$K = -DE_0 \cos \omega t.$$

Equations for the coefficients in the interaction picture

$$\begin{aligned} i\hbar\dot{C}_1 &= -d_{11}(E_0 \cos \omega t)C_1 - d_{12}e^{i\omega_{12}t}(E_0 \cos \omega t)C_2, \\ i\hbar\dot{C}_2 &= -d_{21}(E_0 \cos \omega t)e^{i\omega_{21}t}C_1 - d_{22}(E_0 \cos \omega t)C_2. \end{aligned}$$

The parity selection rule (Laporte) sets:  $d_{11} = d_{22} = 0$ . Also because the dipole moment operator is self-adjoint, the matrix elements  $d_{21} = d_{12}^*$ .

Notation  $d_{21} := d$ ,  $\omega_{21} = \omega_0 \implies \omega_{12} = -\omega_0$ .

# RWA: Rotating Wave Approximation

$$i\dot{C}_1 = -\frac{d^*E_0}{2\hbar}(e^{i\omega t} + e^{-i\omega t})e^{-i\omega_0 t}C_2 = -\frac{d^*E_0}{2\hbar}(e^{i(\omega-\omega_0)t} + \underline{e^{-i(\omega+\omega_0)t}})C_2,$$

$$i\dot{C}_2 = -\frac{dE_0}{2\hbar}(e^{i\omega t} + e^{-i\omega t})e^{+i\omega_0 t}C_1 = -\frac{dE_0}{2\hbar}(\underline{e^{i(\omega+\omega_0)t}} + e^{i(\omega_0-\omega)t})C_1.$$

Close to resonance, on the right hand side we have resonant exponentials with  $\pm(\omega_0 - \omega) = \pm\Delta \ll \omega$ , and "antiresonant ones" with  $\pm(\omega_0 + \omega) \approx 2\omega$ : these are underlined.

We keep the resonant terms while omit the antiresonant ones because on integration

$$e^{\pm i2\omega t} / 2\omega \ll e^{\pm i\Delta t} / \Delta$$

and also because  $e^{\pm i2\omega t}$  oscillate rapidly when compared to  $e^{\pm i\Delta t}$  (see Lectures 6 and 7 on the perturbative approach).

This is the *rotating wave approximation* abbreviated as RWA, valid with a great accuracy close to resonance.

# Rabi frequency

With the RWA, we have:

$$\begin{aligned}\dot{C}_1 &= i\frac{d^*E_0}{2\hbar}e^{i\omega t}e^{-i\omega_{21}t}C_2 = i\frac{d^*E_0}{2\hbar}e^{-i\Delta t}C_2 = i\frac{\Omega_r^*}{2}e^{-i\Delta t}C_2, \\ \dot{C}_2 &= i\frac{dE_0}{2\hbar}e^{-i\omega t}e^{i\omega_{21}t}C_1 = i\frac{dE_0}{2\hbar}e^{i\Delta t}C_1 = i\frac{\Omega_r}{2}e^{i\Delta t}C_1.\end{aligned}\quad (\text{RabiEqs})$$

Here we have also introduced

$$\frac{dE_0}{\hbar} =: \Omega_r \quad \text{the resonant Rabi frequency,} \quad (\text{Rabifr})$$

which is a *key concept* in coherent resonant optics.



# Rabi equations

$$\begin{aligned}\dot{C}_1 &= i\frac{\Omega_r^*}{2}e^{-i\Delta t}C_2, \\ \dot{C}_2 &= i\frac{\Omega_r}{2}e^{i\Delta t}C_1.\end{aligned}\tag{RabiEqs}$$

As we could neglect antiresonant terms (RWA), we obtained that the amplitudes  $C_i$  vary slowly compared to the phase factor  $e^{\pm i\omega t}$ , as their derivatives are determined by  $\Omega_r$ , i.e. by  $E_0(t)$ , which is the *slowly varying amplitude* of the external field.

*Problem:* Check the dimension of  $\Omega_r = \frac{dE_0}{\hbar}$ .

Other notations: Resonant Rabi frequency is sometimes denoted by  $\Omega_0$  (although  $\Omega_0$  has another meaning in quantum optics), by  $\Omega_1$  [1], or by  $\mathcal{R}_0$  [2].

## Phase shifted amplitudes $b$

The system conserves  $|C_1|^2 + |C_2|^2$ , which is set to be

$$|C_1|^2 + |C_2|^2 = 1, \quad (\text{probsum})$$

as, by assumption, we have only these two levels populated.

*Problem: Prove that (probsum) is valid at all times, assuming it holds initially.*

*Problem: Show that for a circularly polarized external field, when  $\mathbf{E} = E_0(\mathbf{x} \cos \omega t + \mathbf{y} \sin \omega t)$  RWA is not necessary, the Rabi equations are exact for a transition between states with magnetic quantum numbers  $m=0$  and  $m=1$ , e.g.  $2p$  and  $1s$ .*

A somewhat simpler set of equations can be obtained if we use new (also slowly varying) amplitudes:

$$b_1(t) := C_1(t)e^{i\Delta t/2} \quad \text{and} \quad b_2(t) := C_2e^{-i\Delta t/2},$$

in terms of which the wave function is expressed as

$$|\Psi(t)\rangle = b_1(t)e^{-i\Delta t/2}e^{-i\frac{\epsilon_1}{\hbar}t} |u_1\rangle + b_2e^{i\Delta t/2}e^{-i\frac{\epsilon_2}{\hbar}t} |u_2\rangle.$$

## Equation for the amplitudes $b$

From the equations for  $C_1$  and  $C_2$  in the RWA we obtain

$$\begin{aligned}\dot{b}_1 &= i\frac{\Delta}{2}b_1 + i\frac{\Omega_r^*}{2}b_2, \\ \dot{b}_2 &= -i\frac{\Delta}{2}b_2 + i\frac{\Omega_r}{2}b_1,\end{aligned}$$

or

$$\frac{d}{dt} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{i}{2} \begin{pmatrix} \Delta & \Omega_r^* \\ \Omega_r & -\Delta \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

The system of equations above can be easily solved by taking the second derivative of say the second equation, substituting  $\dot{b}_1$  from the first one and then seeking the solution in the form  $b_2 = e^{\lambda t}$ . Another more standard way is looking for the "vector" solution of  $\dot{\mathbf{b}} = \mathbf{M}\mathbf{b}$ , with  $\mathbf{b} = \mathbf{b}(0)e^{\lambda t}$ , yielding the eigenvalue equation  $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$ .

# Eigenfrequencies

The solutions are :

$$\lambda = \pm \frac{i}{2} \sqrt{|\Omega_r|^2 + \Delta^2}.$$

By introducing

$$\Omega := \sqrt{\Delta^2 + |\Omega_r|^2},$$

The general solution is a superposition of the solutions with both eigenfrequencies yielding

$$b_2 = (b_{2+} e^{i\Omega t/2} + b_{2-} e^{-i\Omega t/2}).$$

We look here for the solution corresponding to the initial condition

$$b_1(0) = 1, \quad b_2(0) = 0. \quad (\text{initial})$$

## Solution for the atom initially in the lower state

$$b_2 = i \frac{\Omega_r}{\Omega} \sin \frac{\Omega t}{2},$$

$$b_1 = \left( \cos \frac{\Omega t}{2} + i \frac{\Delta}{\Omega} \sin \frac{\Omega t}{2} \right),$$

giving  $C_1(t) = b_1(t)e^{-i\Delta t/2}$  and  $C_2(t) = b_2(t)e^{i\Delta t/2}$ .

*Problem: Check the sum of absolute value squares.*

The probability of finding the system in the upper state

$$|C_2|^2 = |b_2|^2 = \frac{|\Omega_r|^2}{|\Omega_r|^2 + \Delta^2} \sin^2 \frac{\Omega t}{2} = \frac{|\Omega_r|^2}{|\Omega_r|^2 + \Delta^2} \frac{1}{2} (1 - \cos \Omega t)$$

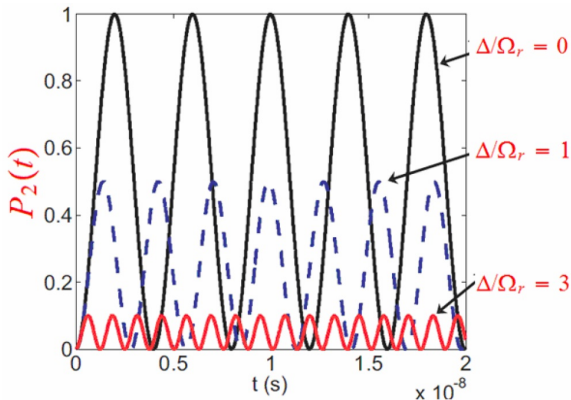
oscillates between 0 and  $\frac{|\Omega_r|^2}{|\Omega_r|^2 + \Delta^2}$  with the frequency of  $\Omega$ . This is the optical Rabi frequency belonging to the detuning  $\Delta$ , named after I. I. Rabi, who derived this result for magnetic resonance in 1937, and used it to determine the magnetic moments of atomic nuclei.

# The general solution of the Rabi problem

*Problem: Show that the solution for the  $b_i$ -s for arbitrary initial conditions  $b_1(0)$  and  $b_2(0)$  can be written as:*

$$\begin{pmatrix} b_2(t) \\ b_1(t) \end{pmatrix} = \begin{pmatrix} \cos \frac{\Omega t}{2} - i \frac{\Delta}{\Omega} \sin \frac{\Omega t}{2} & i \frac{\Omega_r}{\Omega} \sin \frac{\Omega t}{2} \\ i \frac{\Omega_r^*}{\Omega} \sin \frac{\Omega t}{2} & \cos \frac{\Omega t}{2} + i \frac{\Delta}{\Omega} \sin \frac{\Omega t}{2} \end{pmatrix} \begin{pmatrix} b_2(0) \\ b_1(0) \end{pmatrix}.$$

## Rabi oscillations



I. Rabi 1898-1988

1937



1944

The resonance is very sharp, as usually  $\Omega_r \simeq 10^{-5}\omega$ , thus  $\Delta = \Omega_r$  means  $\Delta/\omega \simeq 10^{-5}$ .

## Rabi oscillations II

On exact resonance  $\omega = \omega_0$ ,  $\Delta = 0$ , the Rabi frequency  $\Omega = \Omega_r = dE_0/\hbar$  is just the *resonant Rabi frequency*. In that case

$$|b_1|^2 = \frac{1}{2}(1 + \cos \Omega_r t) = \cos^2 \frac{\Omega_r t}{2},$$

$$|b_2|^2 = |C_2|^2 = \frac{1}{2}(1 - \cos \Omega_r t) = \sin^2 \frac{\Omega_r t}{2}$$

and  $|b_2|^2$  reaches the value 1: there is exact inversion periodically with frequency  $\Omega_r$ , called Rabi flopping.

$$\text{While for } \Omega_r t \ll 1, \quad |C_2|^2 = \sin^2 \frac{\Omega_r}{2} t \approx \frac{\Omega_r}{4} t^2 = \frac{d^2 E_0^2}{4\hbar^2} t^2$$

which is the result we obtained in the previous lecture with perturbation theory, assumed to be valid for short times.



## Some specific pulses

Assume that the field amplitude  $E_0$  is constant, and it is tuned to resonance.

According to  $|C_2|^2 = \sin^2 \frac{\Omega_r}{2} t$ , the system reaches the upper state exactly when  $\Omega_r t = \pi$ .

A pulse with constant amplitude and duration of  $t = \pi/\Omega_r$  is called a  $\pi$  pulse, that inverts the atom.

If  $t = 2\pi/\Omega_r$ , then the atom returns to its initial ground state, this is a  $2\pi$  pulse.

If  $t = \pi/2\Omega_r$ , the pulse brings the atom into an equal-weight superposition of the ground and excited states, this is a  $\pi/2$  pulse.

If the system starts from the upper level, exactly the same processes happen in the opposite direction, if spontaneous emission and other relaxation effects can be neglected on the time scale  $1/\Omega_r$ .

## Pulse angle and pulse area

Essentially the same results are *valid for time-dependent amplitudes*, if we integrate  $\frac{d}{\hbar}E_0(t)$  the amplitude of the field strength, i.e., the resonant ( $\Delta = 0$ ) Rabi frequency with respect to time, giving:

$$\Theta = \frac{d}{\hbar} \int_{-\infty}^t E_0 dt = \int_{-\infty}^t \Omega_r dt$$

and the result is the *pulse angle* in the previous sense.  $\Theta(\infty)$  is called the *pulse area*, which is characteristic to some extent to the strength of the pulse, but *it is not the intensity* which is proportional to  $E_0^2$ .

We see that on exact resonance a  $\pi$  pulse *can excite an atom exactly to its upper state* if this excitation is so fast, that the atom can be described with the Schrödinger equation for all the time. This is a coherent excitation, the atom does not interact with degrees of freedom, other than the oscillating electric field.

## Notes on coherent interaction

Later we shall consider circumstances with incoherent excitation, and will complete the description with relaxation, as well as with pumping terms. These incoherent terms change the situation significantly.

The quantum theory of resonant interaction was derived and used first by Rabi in his molecular beam method. In that case the levels in question were magnetic spin states in nuclei, and the transition was induced by time-varying magnetic fields.

In case of atomic energy levels the transition is the result of the dipole interaction with a time-dependent electric field. Experiments on the validity of the Rabi model in optics was done in the visible domain by H. Gibbs with Rb atoms in 1973.

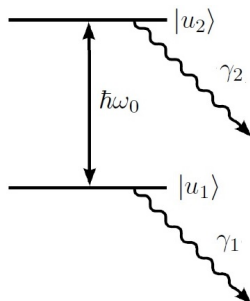
## The necessity of inclusion of relaxation terms

In the coherent interaction between atom and field considered up to now, we have assumed that the atom interacts exclusively with the external field. This is valid, however only for times that are short in comparison of relaxation times. They give the time scales how fast (or slow) is the interaction of the atom with other degrees of freedom, which are always present as an environment, even in perfect vacuum. From now on relaxation and possibly pumping processes, will be taken into account. Relaxation is caused by collisions between the atoms and this will have an important influence on their dynamics. Another damping mechanism – neglected so far – is spontaneous emission.

We will see that absorption and stimulated emission in the sense of Einstein, *which are very much different from the Rabi flopping* obtained in the present lecture, are also the consequences of the relaxation.

## Decay to other levels

The simplest way to introduce relaxation is to add certain damping terms to the amplitude equations above. One of the possible models is shown in the figure:



Energy-level diagram for a two-level atom, showing decay rates  $\gamma_2$  and  $\gamma_1$  for the probabilities  $|b_2|^2$  and  $|b_1|^2$ . Incoherent decay from level 2 to level 1 is neglected.

## Decay to other levels II

The loss of excited-level probability corresponds to an increase of probability of lower-lying levels that we do not consider explicitly, therefore we call this model *Decay to Other Levels*. This model is typical in laser materials.

The equations for the amplitudes in this case can be written with the additional damping coefficients  $\gamma_2$  and  $\gamma_1$  as

$$\dot{b}_1 = \left(-\frac{\gamma_1}{2} + i\frac{\Delta}{2}\right) b_1 + i\frac{\Omega_r^*}{2} b_2, \quad \dot{b}_2 = \left(-\frac{\gamma_2}{2} - i\frac{\Delta}{2}\right) b_2 + i\frac{\Omega_r}{2} b_1.$$

This means that in the absence of the external exciting field when  $E_0 = 0$ , and thus  $\Omega_r = 0$ , the solutions for the level probabilities are

$$|b_1|^2 = |b_1(0)|^2 e^{-\gamma_1 t}, \quad |b_2|^2 = |b_2(0)|^2 e^{-\gamma_2 t}.$$

*This system of equations, however, cannot describe the complexity of the relaxation processes in all its details.*

# Nonunitary evolution

Later we shall also include incoherent pumping, by a field which is different from the resonant field in the Rabi equations.

Pumping can be the result of collisions with other atoms, or relaxation from levels other than the two resonant ones.

The main point here is that the evolution of models with damping (and pumping) are *not unitary* within the two-dimensional subspace of  $|u_1\rangle$  and  $|u_2\rangle$ , and it is impossible to describe them by a self-adjoint Hamiltonian.

*In the presence of damping, the description of the evolution by probability amplitudes loses its advantage. In fact, (as we will see) the method is unable to account for all types of relaxations.*

## Populations

Inclusion of damping requires to treat open quantum systems. Instead of the atomic amplitudes  $C_i$  or  $b_i$  one needs a more complicated concept: the *density matrix*.

In the present context we introduce the elements of the density matrix as *quadratic functions* of the amplitudes.

The probabilities to find the atom in its ground and excited states are

$$|C_1|^2 = C_1 C_1^* = b_1 b_1^* =: \varrho_{11}, \quad |C_2|^2 = C_2 C_2^* = b_2 b_2^* =: \varrho_{22}.$$

The notation  $\varrho_{ii} = |b_i|^2$  refers to a general concept in quantum mechanics: these values are the diagonal elements of the so-called atomic *density matrix*  $\varrho_{ij}$ , introduced by Lev Landau and independently by Janos Neumann in the second half of the 1920's. For our present purposes it is not necessary to give a more general definition of this important concept, we note only that it's use is unavoidable when considering *open systems* in quantum physics, like e.g. the laser. (For a more general setting we refer to the literature.)

The *probabilities*  $\varrho_{ii}$  are called *level populations*.



# Coherences

We shall also introduce the off-diagonal elements of the density matrix for the two-level system by

$$\varrho_{12} = b_1 b_2^* = C_1 C_2^* e^{i\Delta t}, \quad \varrho_{21} = C_2 C_1^* e^{-i\Delta t} = b_2 b_1^* = \varrho_{12}^*.$$

**!Notation:**

The  $\varrho_{ij}$  used here, are actually the elements of the density matrix in the *interaction picture*, denoted usually by  $\tilde{\varrho}_{ij}$  [1], or  $\varrho'_{ij}$  [2]. In those books

$\varrho_{ij} = c_i(t)c_j^*(t) = C_i C_j^* e^{-i\omega_{ij}t}$  stands for the density matrix in the Schrödinger picture!

The off-diagonal elements  $\varrho_{21} = \varrho_{12}^*$  account for how much the system is in the *superposition* of the two stationary states, and they are called *coherences* of the levels in question.

## Dipole moment and the density matrix

The expectation value of the dipole moment in the state

$$|\Psi(t)\rangle = C_1(t)e^{-i\varepsilon_1 t/\hbar} |u_1\rangle + C_2(t)e^{-i\varepsilon_2 t/\hbar} |u_2\rangle$$

$$\begin{aligned} \langle \Psi(t) | D | \Psi(t) \rangle &= C_2 C_1^* \langle u_1 | D | u_2 \rangle e^{-i\omega_0 t} + C_1 C_2^* \langle u_2 | D | u_1 \rangle e^{i\omega_0 t} = \\ &= d(C_2 C_1^* e^{-i\omega_0 t} + C_1 C_2^* e^{i\omega_0 t}) = \\ &= d(\varrho_{21} e^{-i\omega_0 t + i\Delta t} + \varrho_{21}^* e^{i\omega_0 t - i\Delta t}) = d(\varrho_{21} e^{-i\omega t} + \varrho_{21}^* e^{+i\omega t}) \end{aligned}$$

We used here that the static dipole moments vanish:  $\langle u_i | D | u_i \rangle = 0$ ,  $(\varepsilon_2 - \varepsilon_1)/\hbar = \omega_0$ , and  $\omega_0 - \omega = \Delta$ , and assumed  $d = \langle u_2 | D | u_1 \rangle = d^*$ .

We see that in the interaction picture the off-diagonal elements  $\varrho_{21} = \varrho_{12}^*$  give the slowly-varying factor of the dipole moment, induced by the exciting field.

The density matrix is a more complicated object than the amplitudes, but its elements have a more conventional classical meaning, and they are directly measurable quantities.

## The model of decay to other levels

From the equations for the  $b_i$  amplitudes with damping, we get (assuming  $d = d^*$ ), and using  $\dot{\varrho}_{ii} = \dot{b}_i b_i^* + b_i \dot{b}_i^*$ :

$$\dot{\varrho}_{11} = -\gamma_1 \varrho_{11} + i \frac{\Omega_r}{2} (\varrho_{21} - \varrho_{21}^*), \quad (\text{ro11})$$

$$\dot{\varrho}_{22} = -\gamma_2 \varrho_{22} - i \frac{\Omega_r}{2} (\varrho_{21} - \varrho_{21}^*). \quad (\text{ro22})$$

Similarly  $\dot{\varrho}_{12} = \dot{b}_1 b_2^* + b_1 \dot{b}_2^*$  and its conjugate give:

$$\dot{\varrho}_{21} = -\gamma_{21} \varrho_{21} - i \Delta \varrho_{21} - i \frac{\Omega_r}{2} (\varrho_{22} - \varrho_{11}), \quad (\text{ro21})$$

where

$$\gamma_{21} = \frac{1}{2} (\gamma_1 + \gamma_2).$$

If  $\gamma_1 = \gamma_2 = 0$  then adding (ro11) and (ro22) we see that  $\varrho_{11} + \varrho_{22}$  is constant in time, as it should be. If only these two levels are involved in the process, then – as in the previous section – we have to set  $\varrho_{11} + \varrho_{22} = 1$ .

## Phase relaxation I.

Actually, the value of  $\gamma_{21}$  is to be corrected. When we wrote

$$|\Psi(t)\rangle = C_1(t)e^{-i\frac{\varepsilon_1}{\hbar}t} |u_1\rangle + C_2(t)e^{-i\frac{\varepsilon_2}{\hbar}t} |u_2\rangle$$

we assumed fixed values of  $\varepsilon_1$  and  $\varepsilon_2$ . In reality, interactions with other degrees of freedom, or with the environment have the effect to change the values of  $\varepsilon_1$  and  $\varepsilon_2$  randomly by a little amount  $\hbar\delta_1$  and  $\hbar\delta_2$  which are much smaller than the energy difference  $\varepsilon_2 - \varepsilon_1$ , so this perturbation causes random level shifts around the central values  $\varepsilon_1$  and  $\varepsilon_2$ , but no transitions between the levels. The actual state is therefore

$$|\Psi(t)\rangle = C_1(t)e^{-i\frac{\varepsilon_1}{\hbar}t + \delta_1 t} |u_1\rangle + C_2(t)e^{-i\frac{\varepsilon_2}{\hbar}t + \delta_2 t} |u_2\rangle .$$

Obviously, the diagonal elements  $\rho_{ii}$  are not sensitive to such phase disturbances, but  $\rho_{21} = C_2(t)C_1^*(t)e^{-i\Delta t}$  will change to  $\rho_{21} = C_2(t)C_1^*(t)e^{-i\Delta t}e^{i(\delta_1 - \delta_2)t}$ . One cannot follow the actual time variation of the factor  $e^{i\delta t}$  ( $\delta = \delta_1 - \delta_2$ ) explicitly, as  $\delta$  itself changes with time, and in addition these changes are completely random.

## Phase relaxation II.

Therefore, we shall calculate an average of  $\rho_{21}$  over a probability distribution of the  $\delta$ -s, which is assumed to be a Lorentzian:

$$g(\delta) = \frac{\gamma_{ph}}{\pi} \frac{1}{\delta^2 + \gamma_{ph}^2},$$

with a characteristic width  $\gamma_{ph}$ , where the index refers to a distribution of phase differences. For calculating the average  $\rho_{21}$  with respect to  $\delta$  assuming this distribution we use

$$\int_{-\infty}^{\infty} g(\delta) e^{i\delta t} d\delta = \int_{-\infty}^{\infty} \frac{\gamma_{ph}}{\pi} \frac{e^{i\delta t}}{\delta^2 + \gamma_{ph}^2} d\delta = e^{-\gamma_{ph} t}.$$

We obtain a replacement  $\rho_{21} \rightarrow \rho_{21} e^{-\gamma_{ph} t}$  for the off-diagonal elements of the density matrix. This is a so-called phase relaxation of the coherence  $\rho_{21}$ , (and of  $\rho_{12}$ ).

## Phase relaxation III.

The result above shows that the phase relaxation can be modelled simply by adding a term  $-\gamma_{ph}\varrho_{21}$  in the equation for  $\dot{\varrho}_{21}$ . Accordingly we define

$$\gamma := \gamma_{21} + \gamma_{ph}$$

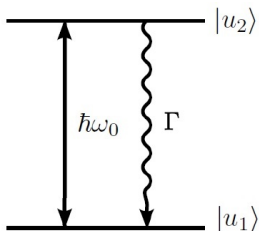
modifying the equation for  $\varrho_{21}$  as

$$\dot{\varrho}_{21} = -\gamma\varrho_{21} - i\Delta\varrho_{21} - i\frac{\Omega_r}{2}(\varrho_{22} - \varrho_{11}). \quad (\text{ro21})$$

The arguments above show the advantage of using the density matrix instead of the state vectors. An equation equivalent to (ro21) together with those for  $\varrho_{11}$  and  $\varrho_{22}$  cannot be constructed from the state vector approach.

## Upper-to-lower-level decay

Another important model with relaxation, where the process cannot be described in terms of probability amplitudes is where the relaxation involves mainly *upper-to-lower-level decay* shown in the figure below, together with the corresponding equations for the density matrix.



$$\dot{\rho}_{22} = -\Gamma \rho_{22} - i\frac{\Omega_r}{2}(\rho_{21} - \rho_{21}^*)$$

$$\dot{\rho}_{11} = \Gamma \rho_{22} + i\frac{\Omega_r}{2}(\rho_{21} - \rho_{21}^*)$$

$$\dot{\rho}_{21} = -\gamma \rho_{21} - i\Delta \rho_{21} - i\frac{\Omega_r}{2}(\rho_{22} - \rho_{11})$$

The probability  $\rho_{22} + \rho_{11} = 1$  is conserved now, since these two levels are the only ones in the problem. It is simply seen, that now even the equations for the populations cannot be rewritten in terms of the amplitudes,  $b_i$ , showing again the necessity to use the density matrix.

## Transversal and longitudinal decay

A notation borrowed from NMR:

$T_1 = 1/\gamma_2 \approx 1/\gamma_1$  or  $T_1 = 1/\Gamma$  called longitudinal decay time

$T_2 = 1/\gamma$  called transversal decay time

For an isolated atom  $\gamma_{ph}$  is negligible, thus  $\gamma \approx (\gamma_1 + \gamma_2)/2$  or  $\gamma = \Gamma$ , and both  $T_1$  and  $T_2$  are in the nanosecond range for a transition in the visible, as stipulated by the constant of spontaneous emission.

In a dense medium like a laser material  $\gamma_{ph}$  is due to collisions with other atoms in a gas, or with phonons in a solid etc., and  $T_2$  is in the picosecond or subpicosecond range, much shorter than for an isolated atom.



# Bloch vector

*Problem: Introduce the following three real variables:*

$$W = \varrho_{22} - \varrho_{11},$$

$$U = 2 \operatorname{Re} \varrho_{21},$$

$$V = -2 \operatorname{Im} \varrho_{21}.$$

*Write down the differential equations for these variables.*

*Show that in the absence of all relaxation terms the length of the so-called Bloch vector  $\mathbf{V} = \{U, V, W\}$  is a constant.*

# Questions

- 1 What conditions verify the two-level resonant approximation?
- 2 What does RWA mean?
- 3 What quantities determine the resonant Rabi frequency?
- 4 What is the phenomenon of Rabi-flopping?
- 5 Into what state does a  $\pi$  pulse drive an initially unexcited atom?
- 6 What is the effect of a  $\pi/2$  pulse?

## Questions (continued)

- 7 What are the physical reasons of relaxation effects?
- 8 What is the mathematical reason why a general relaxation process cannot be described by probability amplitudes (and why one has to use density matrix)?
- 9 What is the physical meaning of the diagonal elements of the density matrix?
- 10 What is the physical meaning of the off-diagonal elements of the density matrix?
- 11 What are the physical consequences of phase relaxation?
- 12 What is the definition of the Bloch vector?

# References

- ① G. W. F. Drake (ed), *Handbook of Atomic, Molecular, and Optical Physics*, Springer (New York) (2006).
- ② P. Meystre and M. Sargent, *Elements of Quantum Optics*, Springer (Berlin, Heidelberg) (2007).