Mathematics

Lecture and practice

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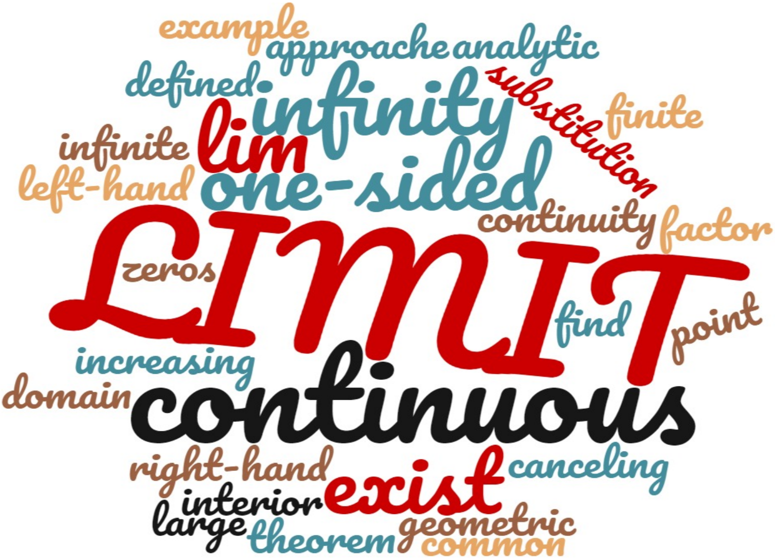
Limit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"Limes"

The word *limes* was used by Latin writers to denote a marked or fortified frontier. This term has been adapted and used by modern historians as an equivalent for the frontiers of the Roman Empire.

Motivating example

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Step 1. Investigation of the domain: the domain of the function is the real numbers except $x = 1$.

Step 2. We substitute some values below and above 1.

x	0	0.5	0.9	0.99	0.999	0.999999	...	1
$f(x)$	1	1.5	1.9	1.99	1.999	1.999999	...	X

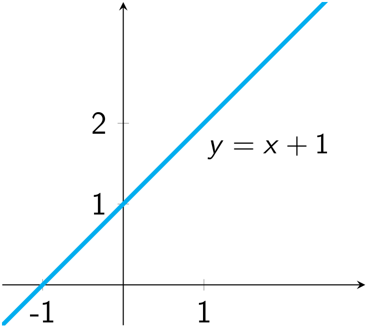
1	...	1.000001	1.0001	1.001	1.01	1.1	1.5	2
X	...	2.000001	2.0001	2.001	2.01	2.1	2.5	3

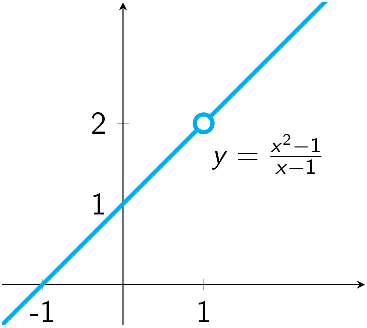
Step 3. We can try to graph the function with computer.

▸ Wolfram Alpha

Step 4. Try to do some mathematics...

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1, \quad (x \neq 1)$$

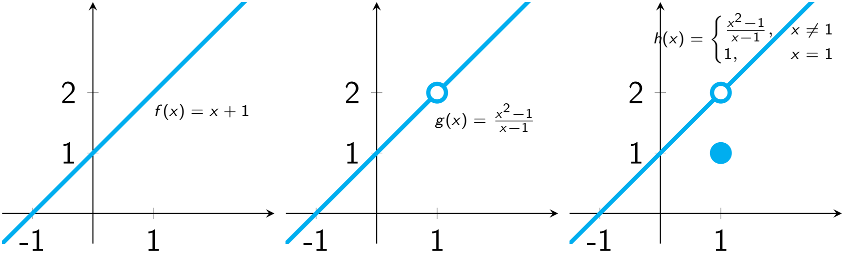




We say that $f(x)$ approaches the limit 2 as x approaches 1, and write

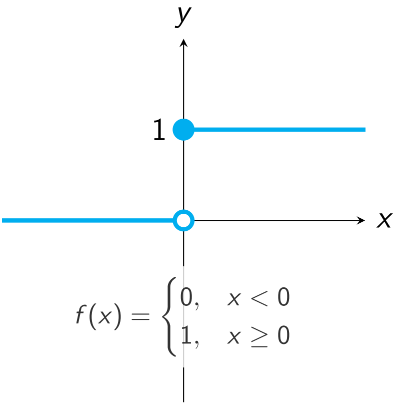
$$\lim_{x\to 1} f(x) = 2 \quad \text{or} \quad \lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

Remark. The limit value does not depend on how the function is defined at x_0.

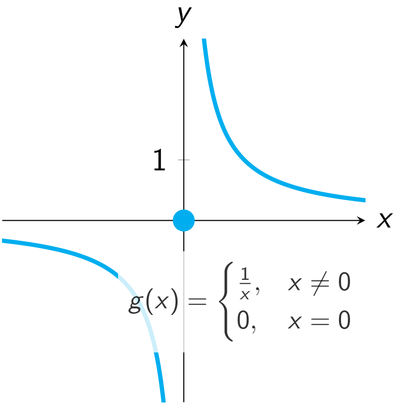


$$\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2$$

What about these?



$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$



$$g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(1) Both of the two functions defined over real numbers (no exclusion).

(2) None of the two functions has limit at $x_0 = 0$.

Limit

Definition

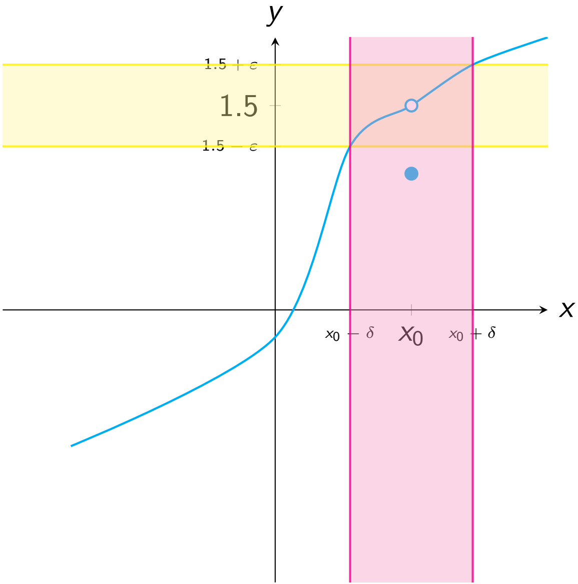
Let $f(x)$ be defined on an open interval about x_0, except at x_0 itself. We say that the limit of $f(x)$ as x approaches x_0 is the number L, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

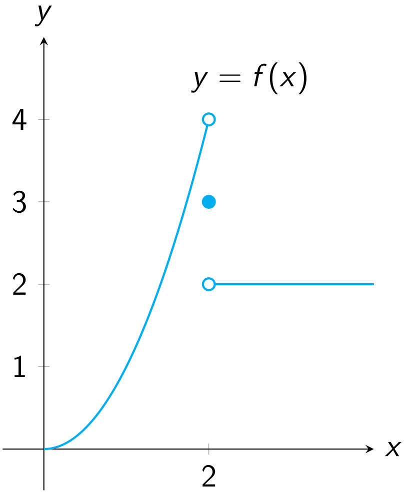
Limit



One-sided limit

- To have a limit L as x approaches c, a function must be defined on *both sides* of c and its values $f(x)$ must approaches c from either side. Because of this, ordinary limits are called **two-sided**.
- If f fails to have a two-sided limit c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

One-sided limit



$$f(2) = 3$$

$$\lim_{x \to 2^+} f(x) = 2$$

$$\lim_{x \to 2^-} f(x) = 4$$

One-sided limit

Definition

If $f(x)$ is defined on an interval (x_0, c), where $x_0 < c$ and approaches arbitrarily close to L as x approaches x_0 from within that interval, then f has **right-hand limit** L at x_0.

$$\lim_{x \to x_0^+} f(x) = L$$

Definition

If $f(x)$ is defined on an interval (a, x_0), where $a < x_0$ and approaches arbitrarily close to M as x approaches x_0 from within that interval, then f has **left-hand limit** M at x_0.

$$\lim_{x \to x_0^-} f(x) = M$$

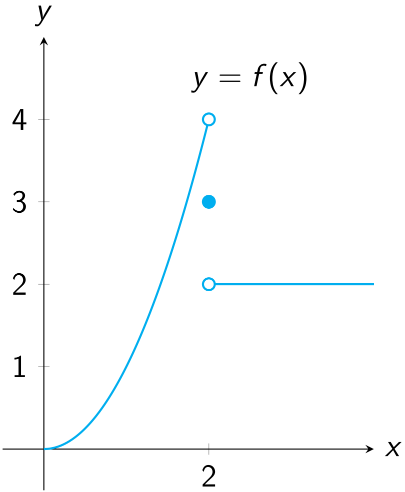
Connection between one- and two-sided limits

Theorem

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = L \text{ and } \lim_{x\to c^-} f(x) = L.$$

One-sided limit

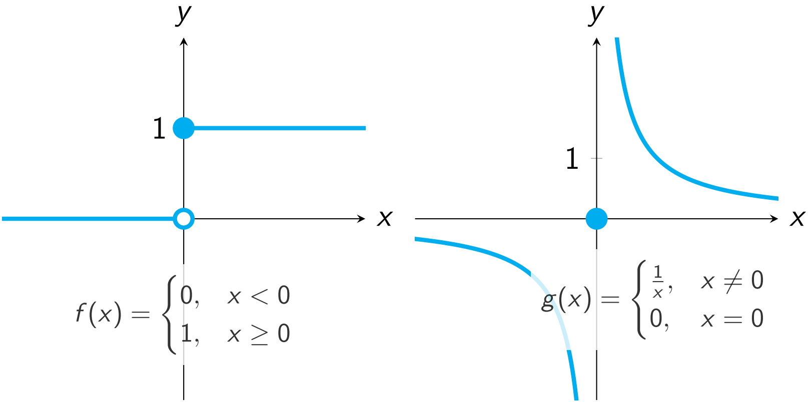


$$\lim_{x \to 2^+} f(x) = 2$$

$$\lim_{x \to 2^-} f(x) = 4$$

$\lim\limits_{x \to 2} f(x)$ doesn't exist

What about these?

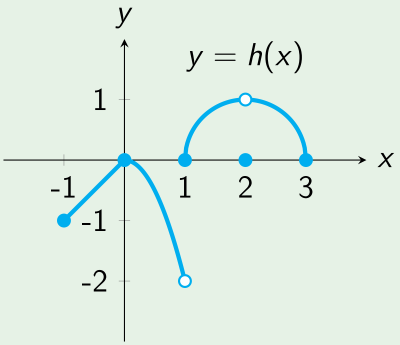


(1) $f(0) = 1, \ \lim\limits_{x \to 0^-} f(x) = 0, \ \lim\limits_{x \to 0^+} f(x) = 1$

(2) $g(0) = 0, \ \lim\limits_{x \to 0} g(x) = ?$

One-sided limit

Exercise



Find the one- and two-sided limits at $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.

Finding limit algebraically

By substitution

Exercise

Find the following limits.

(a) $\lim\limits_{x\to 2} 4 =$

(b) $\lim\limits_{x\to -13} 4 =$

(c) $\lim\limits_{x\to 3} x =$

(d) $\lim\limits_{x\to 2}(5-2x) =$

(e) $\lim\limits_{x\to -1} \dfrac{x^3+4x^2-3}{x^2+5} =$

(f) $\lim\limits_{x\to -2} \dfrac{3x+4}{x+5} =$

Finding limit algebraically

(Creating and) canceling a common factor

Exercise

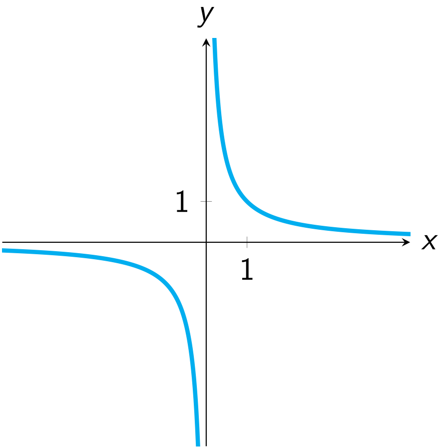
Find the following limits.

(a) $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x^2-x} =$

(b) $\lim\limits_{x\to 0} \dfrac{x^2(x^2-3x+2)}{x^2+x} =$

(c) $\lim\limits_{x\to -2} \dfrac{-2x-4}{x^3+2x^2} =$

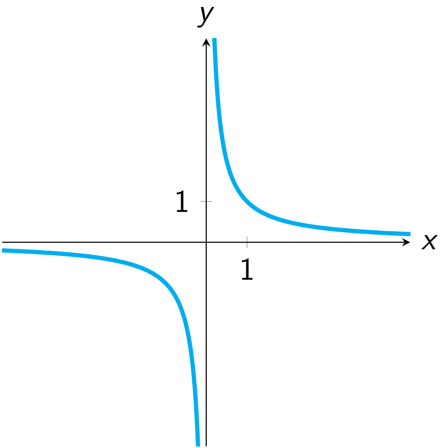
What about $f(x) = \frac{1}{x}$?



- As $x \to 0^+$ the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still.

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

What about $f(x) = \frac{1}{x}$?



- As $x \to 0^-$ the values of f become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$.

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Exercises

(1) $\displaystyle\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} =$

(2) $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2-4} =$

(3) $\displaystyle\lim_{x\to 2^+} \frac{x-3}{x^2-4} =$

(4) $\displaystyle\lim_{x\to 2^-} \frac{x-3}{x^2-4} =$

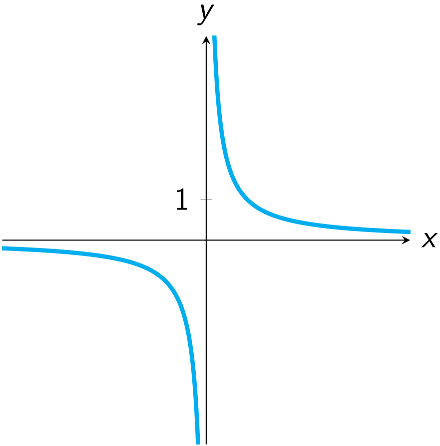
(5) $\displaystyle\lim_{x\to 2} \frac{x-3}{x^2-4} =$

(6) $\displaystyle\lim_{x\to 2} \frac{2-x}{(x-2)^3} =$

Conclusion

Rational functions can behave in various ways near zeros of their denominators.

What about $f(x) = \frac{1}{x}$?



- When x is positive and becomes increasingly large, $1/x$ becomes increasingly small.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

- When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small.

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

Finite limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behaviour of a function when the values in its domain or range outgrow all finite bounds.

Definition

We say that $f(x)$ has the limit L as x approaches infinity (minus infinity) and write

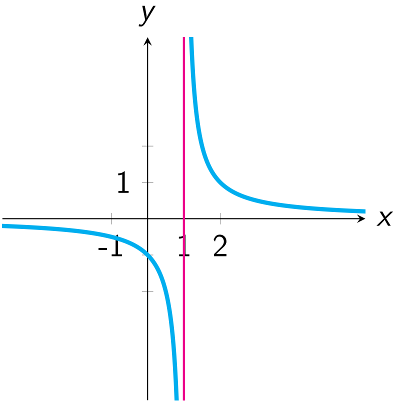
$$\lim_{x\to\infty} f(x) = L, \ \lim_{x\to-\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M (N) such that for all x,

$$x > M, \ (x < N) \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

Exercise

Find the limits of $f(x) = \frac{1}{x-1}$ at $x_0 = 1$ and at infinities.
Geometric solution.



$$\lim_{x\to 1^+} f(x) = \infty$$

$$\lim_{x\to 1^-} f(x) = -\infty$$

$$\lim_{x\to \infty} f(x) = 0$$

$$\lim_{x\to -\infty} f(x) = 0$$

Exercise

Find the limits of $f(x) = \frac{1}{x-1}$ at $x_0 = 1$ and at infinities.
Analytic solution.

Theorem

$$\lim_{x\to\pm\infty} \frac{1}{x} = 0, \quad \lim_{x\to 0^+} \frac{1}{x} = \infty, \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

- $x \to \infty \quad \Longrightarrow \quad x - 1 \to \infty \quad \Longrightarrow \quad \dfrac{1}{x-1} \to 0$
- $x \to -\infty \quad \Longrightarrow \quad x - 1 \to -\infty \quad \Longrightarrow \quad \dfrac{1}{x-1} \to 0$
- $x \to 1^+ \quad \Longrightarrow \quad x - 1 \to 0^+ \quad \Longrightarrow \quad \dfrac{1}{x-1} \to \infty$
- $x \to 1^- \quad \Longrightarrow \quad x - 1 \to 0^- \quad \Longrightarrow \quad \dfrac{1}{x-1} \to -\infty$

Exercises

(1) $\lim\limits_{x\to\infty}\left(5+\dfrac{1}{x}\right)=$

(2) $\lim\limits_{x\to-\infty}\dfrac{\pi\sqrt{3}}{x^2}=$

(3) $\lim\limits_{x\to\infty}\dfrac{5x^2+8x-3}{3x^2+2}=$

(4) $\lim\limits_{x\to-\infty}\dfrac{11x+2}{2x^3-1}=$

(5) $\lim\limits_{x\to\infty}7-\dfrac{8}{x^2}=$

(6) $\lim\limits_{x\to\infty}\dfrac{7x^3}{x^3-3x^2+6x}=$

(7) $\lim\limits_{x\to\infty}\dfrac{2x^3}{5x^2+6x}=$

(8) $\lim\limits_{x\to\infty}\dfrac{2-x^5}{x^3+3x}=$

Exercises

(1) $$\lim_{y\to 2} \frac{y+2}{y^2+5y+6}$$

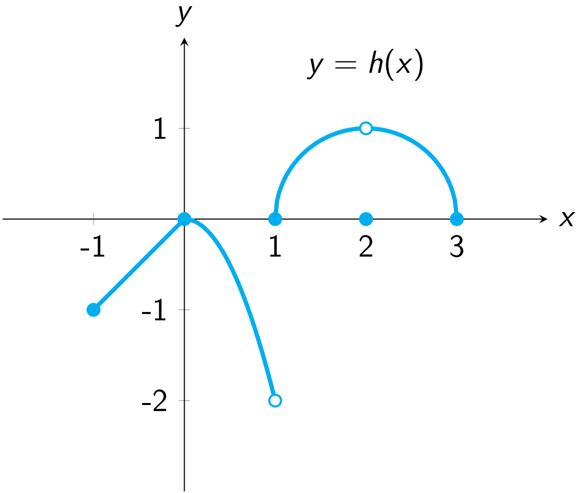
(2) $$\lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}}$$

(3) $$\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2}$$

(4) $$\lim_{x\to \infty} \frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}}$$

(5) $$\lim_{x\to \infty} \frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}$$

Continuity



Continuity

Definition

A function $y = f(x)$ is continuous at an interior point c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Definition

A function $y = f(x)$ is continuous at a left endpoint a or at a right endpoint b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \text{ or } \lim_{x \to b^-} f(x) = f(b), \text{ respectively.}$$

Definition

A function is continuous if it is continuous at every point of its domain.

Continuity

Exercise

Is the function

$$f(x) = \begin{cases} \frac{x^2-4x+4}{x^2+x-6}, & \text{if } x < 2 \\ 0, & \text{if } x = 2 \\ \frac{x^2-3x+2}{x^2-4x+4}, & \text{if } x > 2 \end{cases}$$

continuous at the point $x = 2$?