Mathematics

Lecture and practice

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Differentiation

Motivating example

Example

Next picture shows how a population of fruit flies (Drosophila) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve. Find the average growth rate from day 23 to day 45.

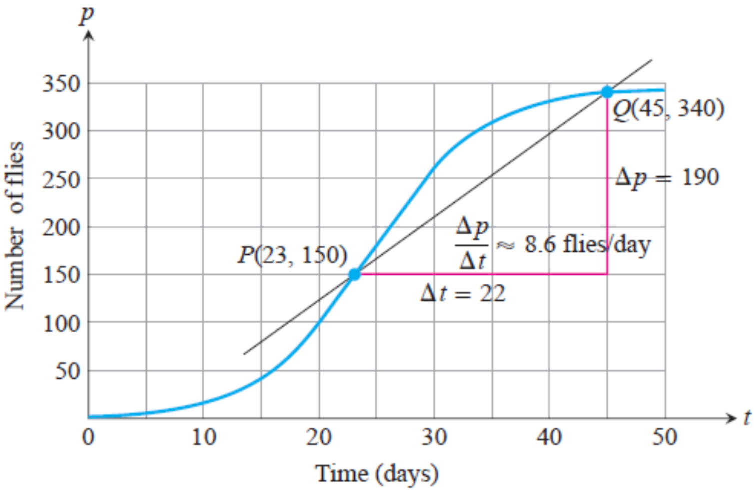


FIGURE 2.2 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p / \Delta t$ of the secant line.

Average rate of change

Definition

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

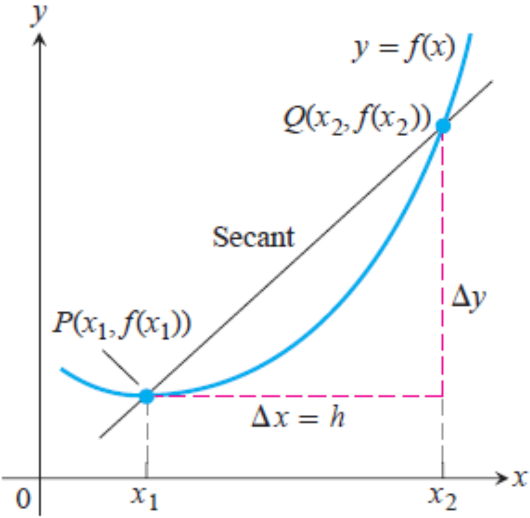
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Geometrically, the rate of change is the slope of the line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Definition

In geometry, a line joining two points of a curve is a **secant** to the curve.

Secant



Motivating example revisited

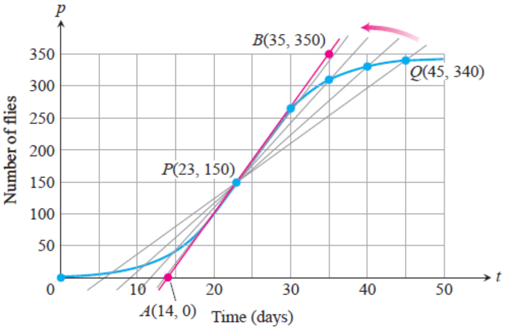
Example

How fast was the number of flies in the population of the previous example growing on day 23?

Motivating example revisited

To answer the question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve.

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340-150}{45-23} \approx 8.6$
(40, 330)	$\frac{330-150}{40-23} \approx 10.6$
(35, 310)	$\frac{310-150}{35-23} \approx 13.3$
(30, 265)	$\frac{265-150}{30-23} \approx 16.4$

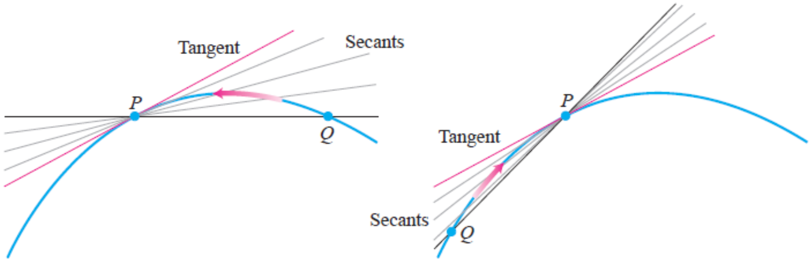


Tangent

What is a tangent of a curve?

Definition

The **tangent** at the point P of a curve is the limit position line of the secants PQ, where Q approaches P.



Tangent and slope

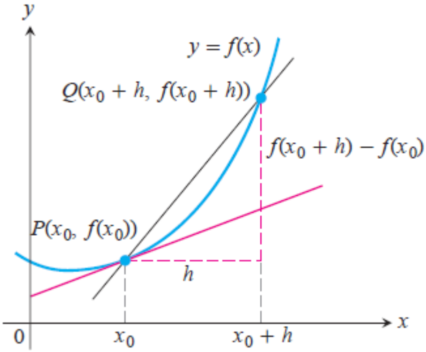


FIGURE 2.67 The slope of the tangent line at P is $\lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$.

Definition

The **slope** of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

(as long as the limit exists).

Definition

The **tangent line** to the curve at P is the line through P with this slope.

Tangent and slope

Finding the tangent to the curve $y = f(x)$ at (x_0, y_0):

(1) Calculate $f(x_0)$ and $f(x_0 + h)$.

(2) Calculate the slope

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

(3) If the limit exists, find the tangent line as $y = y_0 + m(x - x_0)$.

Exercise

Give the tangent to the curve $y = \frac{1}{x}$ at the point $P(2, \frac{1}{2})$.

Derivative function

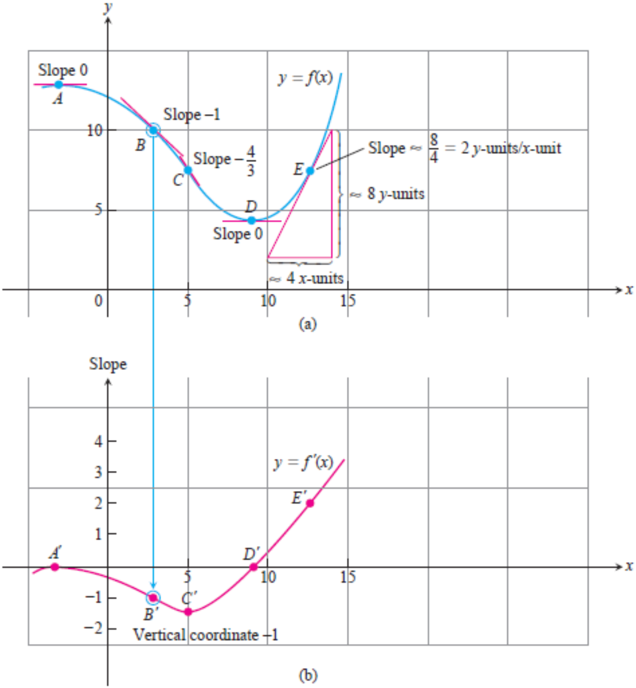
The derivative function $f'(x)$ measures the slope of the tangent line of f at the point x.

Definition

The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. If f' exists at a particular x, we say that f is differentiable (has a derivative) at x.



y
Slope 0
A
Slope −1
10
B
Slope $-\frac{4}{3}$
C
5
D
Slope 0
y = f(x)
E
Slope ≈ $\frac{8}{4}$ = 2 y-units/x-unit
≈ 8 y-units
≈ 4 x-units
0
5
10
15
x
(a)
Slope
4
3
2
1
−1
−2
y = f′(x)
E′
A′
D′
B′
C′
Vertical coordinate −1
5
10
15
x
(b)

Alternative notations and definition

$$f'(x) = y' = \frac{d}{dx}f(x) = \frac{df}{dx} = \lim_{z\to x}\frac{f(z)-f(x)}{z-x}$$

Exercise

Give the derivative of $f(x) = \frac{x}{x-1}$.

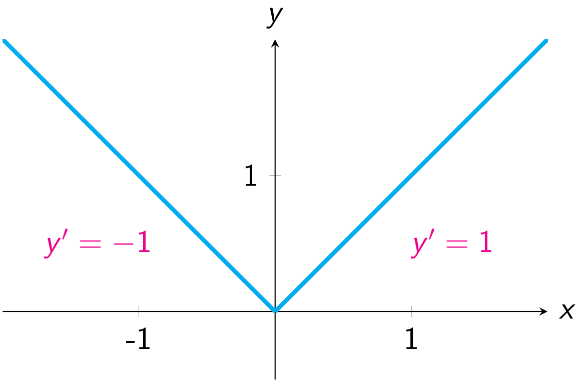
$$f(x+h) = \frac{x+h}{x+h-1}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\frac{x+h}{x+h-1}-\frac{x}{x-1}}{h} = \frac{1}{h}\cdot\frac{(x+h)(x-1)-x(x+h-1)}{(x+h-1)(x-1)} \\ &= \frac{1}{h}\cdot\frac{-h}{(x+h-1)(x-1)} = \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2} \end{aligned}$$

Example

Example

$y = |x|$ is not differentiable at $x_0 = 0$



Example

Example

$y = |x|$ is not differentiable at $x_0 = 0$

Right-hand derivative of $|x|$ at zero:

$$\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

Left-hand derivative of $|x|$ at zero:

$$\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$$

Example

Example

$y = \sqrt{x}$ is not differentiable at $x_0 = 0$

Right-hand derivative of $\sqrt{x}$ at zero:

$$\lim_{h\to 0^+} \frac{\sqrt{0+h}-\sqrt{0}}{h} = \lim_{h\to 0^+} \frac{\sqrt{h}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty$$

Differentiation rules

Theorem (Derivative of a constant function)

If f has the constant value $f(x) = c$, then

$$f'(x) = \frac{d}{dx}c = 0.$$

Theorem (Derivative of a power function)

If n is an integer, then

$$\frac{d}{dx}x^n = n \cdot x^{n-1}.$$

Differentiation rules

Theorem (Constant multiple rule)

If f is a differentiable function of x and c is a constant, then

$$(c \cdot f)' = c \cdot f'.$$

Theorem (Derivative sum rule)

If f and g are differentiable functions of x then their sums $f + g$ is differentiable at every point where f and g are both differentiable. At such points,

$$(f + g)' = f' + g'.$$

Differentiation rules

Theorem (Derivative product rule)

If f and g are differentiable functions of x then their product $f \cdot g$ is differentiable at every point where f and g are both differentiable. At such points,

$$(f \cdot g)' = f' \cdot g + f \cdot g'.$$

Theorem (Derivative quotient rule)

If f and g are differentiable functions of x then their quotient f/g is differentiable at every point where f and g are both differentiable and $g(x) \neq 0$. At such points,

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}.$$

Differentiation rules

Theorem (Derivative of a composite function)

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$[f(g(x))]' = f'(g(x)) \cdot g'(x).$$

Derivatives of elementary functions

Function $f(x)$	Derivative $f'(x)$
c	0
x^α	$\alpha \cdot x^{\alpha-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\log x \ (\ln x)$	$1/x$

Exercises

Determine the derivatives of the following functions.

$$3x^{10} \qquad 9x^4 + 6x^2 + 1 \qquad \frac{3x+5}{2} \qquad 3x^5 - \frac{x}{2} + \frac{2}{x} \qquad \sqrt{x} - \frac{1}{x^2}$$

$$x^3\sqrt{x} \qquad \sqrt[3]{x}\cdot(x^2+2), \qquad (3x^2 - x + 1)(3 - 3x^5 + x^7)$$

$$\frac{1}{x-1} \qquad \frac{3x-1}{1-x} \qquad \frac{1-x^2-3x^5}{x^2+5x-1} \qquad \frac{\sqrt{x}-2x}{3x^2-\sqrt[5]{x}}$$

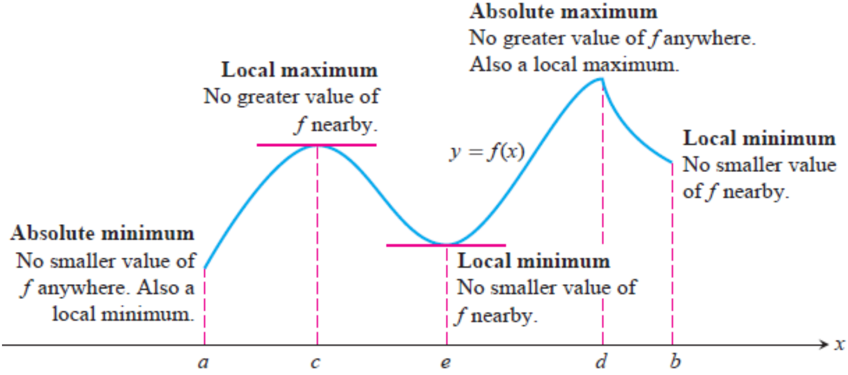
$$(3x^2+1)^2 \qquad (7+3x^8)^{12} \qquad \frac{1}{(1-2x)^3} \qquad 4(5x^3-x^4)^7$$

$$\sqrt[3]{1-x} \qquad \sqrt[4]{x^3-2x^2} \qquad \sqrt{x^2+1}\cdot\sqrt[4]{2x^2-3x^3}$$

Local extreme values

Definition

A function f has a local maximum (minimum) value at an interior point c of its domain if $f(x) \leq f(c)$ ($f(x) \geq f(c)$) for all x in some open interval containing c.



Finding extrema

Theorem

If f has a local extreme value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

The function f can possibly have an extreme value (local or global) at

- interior points where $f' = 0$;
- interior points where f' is undefined;
- endpoints of the domain of f.

Definition

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Finding extrema

How to find the absolute extrema of a continuous function f on a finite closed interval

(1) Evaluate f at all critical points and endpoints.

(2) Take the largest and smallest of these values.

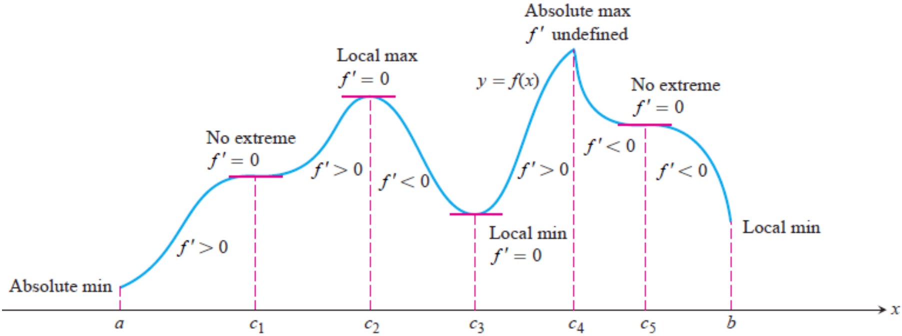
Exercise

Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Exercise

Find the absolute extrema values of $f(x) = x^{2/3}$ on $[-2, 3]$.

Monotonicity



Monotonicity

Definition

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in L.

(1) If $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.

(2) If $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

Theorem

Suppose the f is continuous on $[a, b]$ and differentiable on (a, b).

- *If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.*
- *If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.*

Monotonicity

Theorem (First derivative test for local extrema)

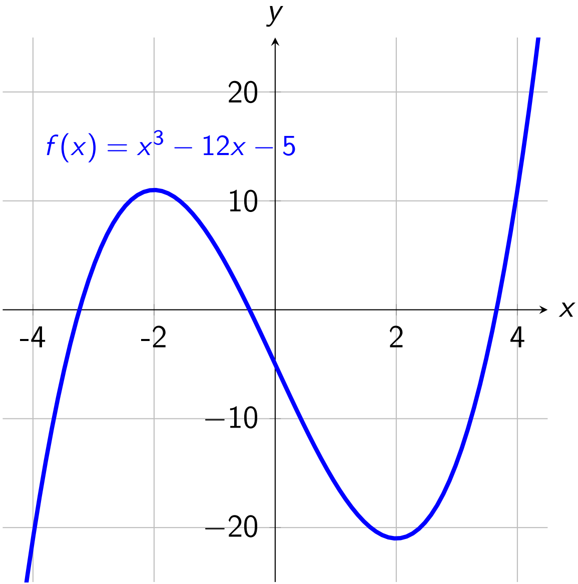
Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. *if f' changes from negative to positive at c, then f has local minimum at c;*
2. *if f' changes from positive to negative at c, then f has local maximum at c;*
3. *if f' does not change sign at c, then f has no local extremum at c.*

Exercise

Investigate the monotonicity of the functions $f(x) = x^3 - 12x - 5$ and $g(x) = x^{1/3}(x - 4)$.

Plot



$g(x) = x^{1/3}(x-4)$

$$g(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$$

Derivative:

$$g'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-1) = \frac{4(x-1)}{3x^{2/3}}$$

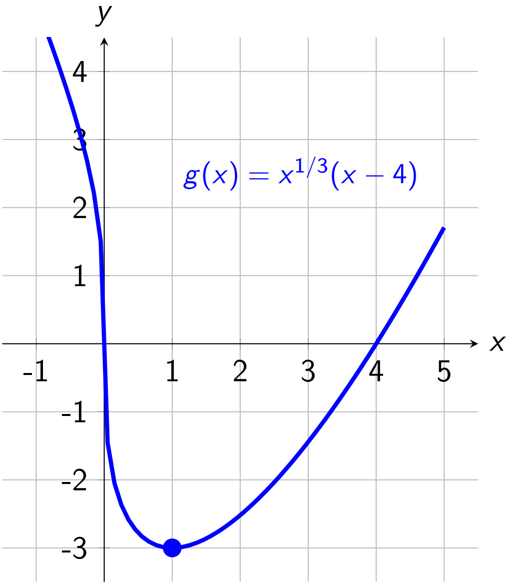
Critical points:

- At $x = 1$, $g'(x) = 0$.
- At $x = 0$, $g'(x)$ is undefined.

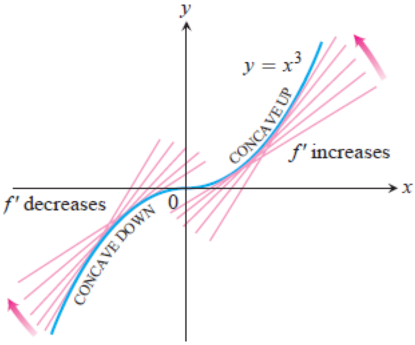
Local extrema may occur at these points.

	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x$
g'	$-$	undef	$-$	0	$+$
g	$\searrow$		$\searrow$	MIN	$\nearrow$

Plot



Concavity



Definition

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

Definition

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

Concavity

Theorem (The second derivative test for convexity)

Let $y = f(x)$ be twice-differentiable on an interval I.

1. *If $f'' > 0$ on I, the graph of f over I is concave up.*
2. *If $f'' < 0$ on I, the graph of f over I is concave down.*

Sketching a graph

Strategy for graphing $y = f(x)$.

1. Identify the domain of f.
2. Find $f'(x)$ and the critical points.
3. Find where the curve is increasing and where it is decreasing and where it has local extrema.
4. Find $f''(x)$ and the critical points.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Plot key points, such as the intercepts and the points found in step 2-5, and sketch the curve.

Exercise

Sketch the graph of the function $f(x) = x^4 - 4x^3 + 10$ and $g(x) = \frac{(x+1)^2}{1+x^2}$.

$g(x) = \frac{(x+1)^2}{1+x^2}$

1. The domain of the function is $(-\infty, \infty)$.

2. $g'(x) = \dfrac{(1+x^2)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2} = \dfrac{2(1-x^2)}{(1+x^2)^2}$
   Critical points: $x = 1$, $x = -1$.

3. Table for first derivative:

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x$
g'	$-$	0	$+$	0	$-$
g	$\searrow$	MIN	$\nearrow$	MAX	$\searrow$

$g(x) = \frac{(x+1)^2}{1+x^2}$

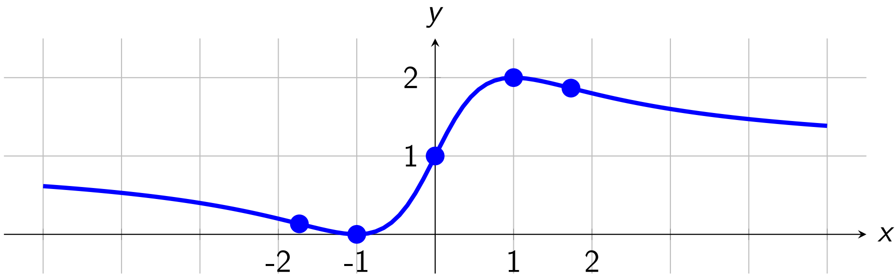
4. $g''(x) = \ldots = \dfrac{4x(x^2-3)}{(1+x^2)^3}$
   Critical points: $x = 0$, $x = \sqrt{3}$, $x = -\sqrt{3}$.

5. Table for second derivative:

	$x <$	$-\sqrt{3}$	$< x <$	0	$< x <$	$\sqrt{3}$	$< x$
g''	$-$	0	$+$	0	$-$	0	$+$
g	$\frown$	IP	$\smile$	IP	$\frown$	IP	$\smile$

6. Intersection with y-axis: $g(0) = 1 \implies (0,1)$.
   Intersection with x-axis: solve $g(x) = 0 \implies x = -1 \implies (-1,0)$.

$g(x) = \frac{(x+1)^2}{1+x^2}$



Physics: motion along a line: displacement, velocity, speed, acceleration, and jerk

Suppose that an object is moving along a coordinate line (say an s-axis) so that we know its position s on that line as a function of time t:

$$s = f(t).$$

Definition

The **displacement** of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t).$$

Physics: motion along a line: displacement, velocity, speed, acceleration, and jerk

Definition

The **average velocity** of the object over the time interval from t to $t + \Delta t$ is

$$v = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t}$$

Definition

Velocity (instantaneous velocity) is the derivative of position respect to time. If a body's position at time t is $s(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = s'(t).$$

Physics: motion along a line: displacement, velocity, speed, acceleration, and jerk

Definition

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s(t)$, then the body's acceleration at time t is

$$a(t) = v'(t) = s''(t).$$

Definition

Jerk is the derivative of acceleration with respect to time. If a body's position at time t is $s(t)$, then the body's acceleration at time t is

$$j(t) = a'(t) = v''(t) = s'''(t).$$

Physics: motion along a line: displacement, velocity, speed, acceleration, and jerk

Example

A particle is moving along a horizontal line with position function

$$s(t) = 2t^3 - 14tt^2 + 22t - 5, \ (t \geq 0).$$

Find the velocity and acceleration, and describe the motion of the particle.

Applied optimization problems

Example

An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Example

Suppose that $r(x) = 9x$ (revenue) and $c(x) = x^3 - 6x^2 + 15x$ (cost), where x represents thousands of units. Is there a production level that maximizes profit? If so, what is it?

Mean Value Theorem

Theorem (Rolle's theorem)

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number $c \in (a, b)$ at which $f'(c) = 0$.

Geometrical meaning:

Drawing the graph of a function gives strong geometric evidence that between any two points where a differentiable function crosses a horizontal line there is at least one point on the curve where the tangent is horizontal.

Mean Value Theorem

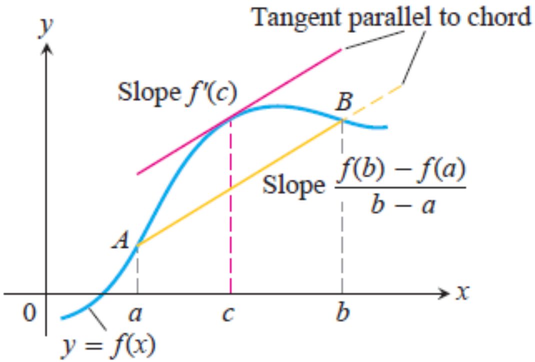
Theorem (The Mean Value Theorem)

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

The Mean Value Theorem, which was first stated by Joseph-Louis Lagrange, is a slanted version of Rolle's Theorem. There is a point c where the tangent is parallel to chord AB.

Mean Value Theorem

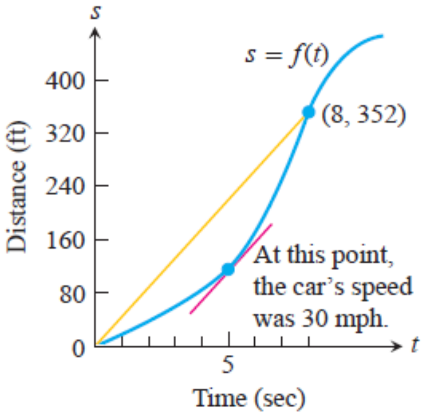


The Mean Value Theorem, which was first stated by Joseph-Louis Lagrange, is a slanted version of Rolle's Theorem. There is a point c where the tangent is parallel to chord AB.

Mean Value Theorem

Example

If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. At some point during the acceleration, the Mean Value Theorem says, the speedometer must read exactly 30 mph (44 ft/sec).



Newton—Rapson method

This is a numerical method, which is a technique to approximate the solution to an equation $f(x) = 0$.

Procedure

(1) Guess a first approximation to a solution of the equation $f(x) = 0$. (A graph of $y = f(x)$ may help.)

(2) Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ if } f'(x_n) \neq 0.$$

Newton—Rapson method

Example

Estimate the value of $\sqrt{2}$.

- We have to estimate the zero of the polynomial $f(x) = x^2 - 2$.
- $f'(x) = 2x$
- Iteration steps:

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}.$$

Newton's method is the method used by most calculators to calculate roots because it converges so fast.