Intersection of Ovals and Unitals in Desarguesian Planes

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- Problem 1 is open in the general case.
- Known results concern the classical case.
- Classical case:
- Oval is an irreducible conic Ω, for q odd the set of all absolute points of an orthogonal polarity π of $PG(2, q^2)$,
- Unital is a Hermitian curve $\mathcal{U}$, i.e. set of all absolute points of a unitary polarity ω of $PG(2, q^2)$.

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Previous results from Segre's work 1965

Unital and oval in permutable position

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- *For q odd, a classical Baer suboval can be the complete intersection of a classical oval and a Hermitian unital.*
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Theorem

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Remark

Proof uses counting arguments, computations with Hermitian forms plus discussions about special equations over a finite field.

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Classification of $\Omega \cap \mathcal{U}$

Theorem (Donati-Durante-Korchmáros, 2009)

In $PG(2, q^2)$ with $q > 3$, the intersection pattern of a classical ovale Ω and a Hermitian unital $\mathcal{U}$ is one of the following.

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(VI) $|\Omega \cap \mathcal{U}| = k$ *with* $k \in \{q, q+1, q+2\}$

(VII) $|\Omega \cap \mathcal{U}| = k$ *with* $(\sqrt{q}-1)^2 + 1 \le k \le (\sqrt{q}+1)^2 + 1$

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- $GF(q^2) = GF(q)(i)$ with

$$\begin{cases} i^2 = s \text{ with nonsquare } s \in GF(q) \text{ when } q \text{ odd;} \\ i^2 + i = \delta \text{ with } \delta \in GF(q),\ \mathrm{Tr}(\delta) = 1, \text{ when } q \text{ even.} \end{cases}$$

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- $\mathcal{U} :=$ Hermitian unital associated to the unitary $\mathbf{U}$.
- For $p > 2$,

$$\mathbf{U} = \begin{pmatrix} a_{11} & a_{12} + b_{12}i & a_{13} + b_{13}i \\ a_{12} - b_{12}i & a_{22} & a_{23} + b_{23}i \\ a_{13} - b_{13}i & a_{23} - b_{23}i & a_{33} \end{pmatrix}$$

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- For $p = 2$,

$$\mathbf{U} = \begin{pmatrix} a_{11} & a_{12} + b_{12}i & a_{13} + b_{13}i \\ a_{12} + b_{12}(i+1) & a_{22} & a_{23} + b_{23}i \\ a_{13} + b_{13}(i+1) & a_{23} + b_{23}(i+1) & a_{33} \end{pmatrix}$$

Proof of Classification $\Omega \cap \mathcal{U}$ for $p > 2$

det((**U**)

- For $p > 2$, $\det(\mathbf{U}) =$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}^2 + sa_{11}b_{23}^2 - a_{22}a_{13}^2 + sa_{22}b_{13}^2 - a_{12}^2a_{33} + 2a_{12}a_{13}a_{23} - 2sa_{12}b_{13}b_{23} + sa_{33}b_{12}^2 + 2sa_{13}b_{12}b_{23} - 2sa_{23}b_{12}b_{13},$$

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$$a_{11}a_{22}a_{33} + a_{11}a_{23}^2 + a_{11}a_{23}b_{23} + \delta a_{11}b_{23}^2 + a_{22}a_{13}^2 + a_{22}a_{13}b_{13} + \delta a_{22}b_{13}^2 + a_{12}^2a_{33} + a_{12}a_{33}b_{12} + a_{12}a_{13}b_{23} + a_{12}a_{23}b_{13} + \delta a_{33}b_{12}^2 + a_{13}a_{23}b_{12} + a_{13}b_{12}b_{23} + \delta b_{12}b_{13}b_{23}$$

Proof of Classification $\Omega \cap \mathcal{U}$

- Let $t = x + yi$, $t \in GF(q^2), x, t \in GF(q)$ and $P_t \in \mathcal{C}$, $t \in GF(q^2)$. Then $P_t \in \mathcal{U} \iff$

$$(1, t^q, t^{2q})\,\mathbf{U} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} = 0.$$

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This condition is equivalent for $p > 2$ to

$$\begin{aligned} e(x,y) = \quad & a_{33}(x^2 - sy^2)^2 + 2(a_{23}x + b_{23}sy)(x^2 - sy^2) + \\ & (a_{22} + 2a_{13})x^2 + 4b_{13}sxy + s(2a_{13} - a_{22})y^2 + \\ & 2a_{12}x + 2b_{12}sy + a_{11} = 0. \end{aligned}$$

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and for $p = 2$ to

$$\begin{aligned} e(x,y) = \quad & a_{33}(x^2 + xy + \delta y^2)^2 + (x^2 + xy + \delta y^2) \\ & (b_{23}x + (a_{23} + b_{23})y) + (a_{22} + b_{13})x^2 + \\ & a_{22}xy + (a_{13} + b_{13} + (a_{22} + b_{13})\delta)y^2 + \\ & b_{12}x + (a_{12} + b_{12})y + a_{11} = 0 \end{aligned}$$

- $\mathcal{E} :=$ curve of equation $e(x,y) = 0$.

- *Let M be the number of points $Q = (x, y)$ in $AG(2, q)$ which lie on $\mathcal{E}$. Then*

$$M = \begin{cases} |\Omega \cap \mathcal{U}| & P_\infty \notin \mathcal{U}; \\ |\Omega \cap \mathcal{U}| - 1 & P_\infty \in \mathcal{U}. \end{cases}$$

Proposition

$M = 0 \iff \mathcal{E}$ *splits into two distinct absolutely irreducible conics defined over a quadratic extension* $GF(q^2)$ *of* $GF(q)$ *having no point in* $PG(2, q)$ *and conjugate to each other over* $GF(q)$.

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- For $p > 2$, this is the case $\iff$

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\begin{aligned}
&a_{11} = \gamma_1^2 - s\gamma_2^2, && a_{12} = \alpha_1\gamma_1 - s\alpha_2\gamma_2,\\
&a_{22} = 2\gamma_1 + \tfrac{1}{2}\left(\alpha_1^2 - s\alpha_2^2 - \tfrac{1}{s}(\beta_1^2 - s\beta_2^2)\right), && a_{23} = \alpha_1, a_{33} = 1,\\
&a_{13} = \tfrac{1}{4}(\alpha_1^2 - s\alpha_2^2 + \tfrac{1}{s}(\beta_1^2 - s\beta_2^2)), && b_{12} = \tfrac{1}{s}(\beta_1\gamma_1 - s\beta_2\gamma_2),\\
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- Example: $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, $\gamma_1 = \gamma_2 = 1$,

$$\mathbf{U} = \begin{pmatrix} 1-s & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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- For $p = 2$, similar results hold.

- $P_\infty \in \Omega \cap \mathcal{U}$ is assumed $\Longrightarrow a_{33} = 0 \Longrightarrow \deg \mathcal{E} \leq 3$.

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- $\mathcal{E}$ is a line (this case cannot actually occur),

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If $\mathcal{E}$ *is an absolutely reducible conic, then* $|\mathcal{U} \cap \Omega|$ *can only assume a few values, namely* $1, 2, q+1, 2q, 2q+1$ *for* $p > 2$*, and* $2, 2q$ *for* $p = 2$*. More precisely, one of the following cases occurs:*

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If $\mathcal{E}$ is an irreducible conic, then

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The method in the above proof may work when either Ω is a translation oval or Segre's oval or $\mathcal{U}$ is a Buekenhout-Metz unital. However the arising plane curve $\mathcal{E}$ may not have low degree.

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The method in the above proof may also work in $PG(r, q^2)$ with $r \geq 3$ when Ω is a rational normal curve and $\mathcal{U}$ is a Hermitian variety.

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- **Given an oval Ω in $PG(2,q)$ with q odd, how many points of another oval Γ in $PG(2,q)$ are external to Ω?**
- It is not hard to find Γ such that either all its points off Ω are external to Δ, or none of its points are.

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- *If* $q \ge 17$ *and* Γ' *consists entirely of points external to* Ω, *then* Ω' *consists entirely of external points of* Γ *or entirely of internal points of* Γ. *Both cases occur.*