Extremal Theorems in Random Discrete Structures

József Balogh

February 2013

Recent trends in combinatorics

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- Classical extremal Theorems in combinatorics.

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- Classical extremal Theorems in combinatorics.
- Is the random sparse variant true?

Recent trends in combinatorics

Example: Bounded degree Trees in graphs

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  If $\delta(G_n) \geq (1 + o(1))n/2$ then $T_n \subset G_n$ for every bounded degree tree.

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  For every $\epsilon > 0, d$ if $p > \frac{d}{\epsilon n} \log \frac{1}{\epsilon}$ then w.h.p. $G(n, p)$ contains every tree T with $|T| < (1 - \epsilon)n$, $\Delta(T) < d$.

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- Balogh, Csaba, and Samotij (2011)
  A.a.s. every subgraph of $G(n, p)$ with minimum degree at least $(1/2 + \epsilon)np$ contains every bounded degree tree with $(1 - \epsilon)n$ vertices, where $p > C(\epsilon)/n$.

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- For what p will a.a.s. $G(n,p)$ contain every bounded degree spanning tree?

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- For what p will a.a.s. $G(n,p)$ contain every bounded degree spanning tree? Johannsen, Krivelevich, Samotij (2012)
  $p = n^{-1/3+o(1)}$ is sufficient.

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Example: Triangle factors in graphs

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  If $\delta(G_{3n}) \geq 2n$ then G_{3n} contains a triangle-factor.

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  For all $\gamma > 0$, there exists C such that if $p \gg ((\log n)/n)^{1/2}$, then a.a.s. every $H \subset G(n; p)$ with $\delta(H) > (2/3 + \gamma)np$ contains a triangle packing that covers all but at most C/p^2 vertices.
- Sudakov, Hao, Lee (2011) C/p^2 is best possible.

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- Sudakov, Hao, Lee (2011) C/p^2 is best possible.
- What about larger cliques?

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Example: K_s-factors in graphs

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- Hajnal, Szemerédi (1970)
  If $\delta(G_{sn}) \geq (s-1)n$ then G_{sn} contains a K_s-factor.

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  If $p \gg n^{-(s-1)/[s(s-1)]}(\log n)^{1/s}$ then w.h.p. $G(sn, p)$ contains a K_s-factor.

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- Balogh, Morris, Samotij (2012+)
  Conlon, Gowers, Samotij, Schacht (2012++)
  For every s if $p \gg n^{-2/(s+1)}$, then a.a.s. every $H \subset G(sn; p)$ with $\delta(H) > [s-1+o(1)]np$ contains a K_s-packing that covers all but $o(n)$ vertices.

Classical extremal Theorems in combinatorics

Example: Turán type Theorems

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- Turán (1941)
  $\mathrm{ex}(n, K_{k+1}) = t_k(n) = (1 - \frac{1}{k} + o(1))\binom{n}{2}$.

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- Erdős, Frankl, Rödl (1986)
  There are $2^{(1+o(1))\cdot \mathrm{ex}(n,K_{k+1})}$ K_{k+1}-free graphs on n vertices.

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  Almost all K_{k+1}-free graphs are k-partite.

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Related Sparse questions:

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Related Sparse questions:

- For what p is the following true?
  $\mathrm{ex}(G(n,p), K_{k+1}) = (1 - \frac{1}{k} + o(1))p\binom{n}{2}$.

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- For what p is the following true?
  $\mathrm{ex}(G(n,p), K_{k+1}) = (1 - \frac{1}{k} + o(1))p\binom{n}{2}$.
- For what $m = m(n)$ is the following true?
  The number K_{k+1}-free graphs with m edges is $\binom{t_k(n)+o(n^2)}{m}$.

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  The number K_{k+1}-free graphs with m edges is $\binom{t_k(n)+o(n^2)}{m}$.
- For what $m = m(n)$ is the following true?
  A.a. K_{k+1}-free graphs with m edges are (almost) k-partite.

Classical Extremal Theorems in Additive Combinatorics

Classical extremal Theorems in combinatorics

Definition

We say that $A \subset [n]$ is (δ, k)-Szemerédi, if every $B \subset A$ with $|B| > \delta|A|$ contains an arithmetic progression of length k. (k-AP).

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Theorem Szemerédi (1975)

For every $\delta > 0, k$, for sufficiently large n, the set $[n]$ is (δ, k)-Szemerédi.

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For every $\delta > 0, k$, for sufficiently large n, the set $[n]$ is (δ, k)-Szemerédi.

Related Sparse questions:

- For what $p = p(\delta, k)$ a p-random subset of $[n]$ is w.h.p. (δ, k)-Szemerédi?

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Related Sparse questions:

- For what $p = p(\delta, k)$ a p-random subset of $[n]$ is w.h.p. (δ, k)-Szemerédi?
- For a given m, how many m-subset of $[n]$ does not contain an k-AP?

Generalizing Szemerédi's theorem

Conjecture (Erdős)

If $R \subseteq \mathbb{N}$ satisfies $\sum_{n \in R} \frac{1}{n} = \infty$, then R contains arbitrarily long APs.

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Generalizing Szemerédi's theorem

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If $R \subseteq \mathbb{N}$ satisfies $\sum_{n \in R} \frac{1}{n} = \infty$, then R contains arbitrarily long APs.

Theorem (Green–Tao [2004])

Let $A \subseteq \mathbb{P}$ be a subset of the primes whose upper density is positive, i.e.,

$$\limsup_{n \to \infty} \frac{|A \cap [n]|}{|\mathbb{P} \cap [n]|} > 0$$

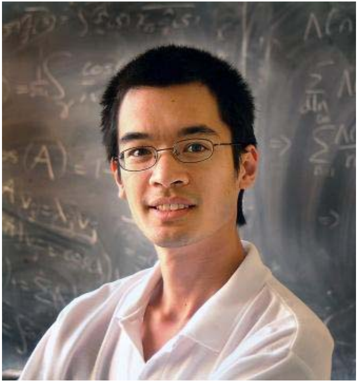
Then A contains arbitrarily long APs.

Generalizing Szemerédi's theorem

Theorem (Green–Tao [2004])

Every subset of the primes with positive upper density contains arbitrarily long APs.

B. Green



T. Tao

Extremal problems – general framework

Framework

Setting:

- A finite set V and a k-uniform hypergraph $\mathcal{H} \subseteq \mathcal{P}(V)$ on V.

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- What does a typical independent set look like?

Questions: (Sparse random variants)

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Questions: (Sparse random variants)

- What is the size of the largest independent set in $G^{(v)}(\mathcal{H}, p)$?

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Extremal problems – general framework

Example: (Turán problem)

- $V(\mathcal{H}) = E(K_n)$,
- $E(\mathcal{H}) =$ edge-sets of copies of K_k in K_n,
- $\mathcal{H}$ is $\binom{k}{2}$-uniform,
- Independent sets in $\mathcal{H}$ $\quad \rightarrow \quad$ K_k-free subgraphs in K_n.

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Example: (Szemerédi's Theorem)

- $V(\mathcal{H}) = \{1, \ldots, n\}$,
- $E(\mathcal{H}) = k$-term APs in $[n]$,
- $\mathcal{H}$ is k-uniform,
- Independent sets in $\mathcal{H}$ $\rightarrow$ k-AP-free subsets of $[n]$.

Sparse random analogues

Metatheorem (Conlon, Gowers; Schacht (2010+++))

'dense' extremal result
$+$
supersaturation
$\Longrightarrow$ 'sparse' extremal result.

Dr D. Conlon

Sir W.T. Gowers

Dr M. Schacht

Sparse random analogues

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'dense' extremal result
$+$
supersaturation
$\Longrightarrow$ 'sparse' extremal result.

Theorem (Conlon, Gowers; Schacht)

For every k and if $p \geq C(k) \cdot n^{-\frac{2}{k+2}}$, then a.a.s.,

$$\mathrm{ex}(G(n,p), K_{k+1}) = \left(1 - \frac{1}{k} + o(1)\right)\binom{n}{2}p.$$

Sparse random analogues

Metatheorem (Conlon, Gowers; Schacht (2010+++))

'dense' extremal result + supersaturation $\implies$ 'sparse' extremal result.

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For every k and if $p \geq C(k) \cdot n^{-\frac{2}{k+2}}$, then a.a.s.,

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Theorem (Conlon, Gowers; Schacht)

For every $k \geq 3$ and $\delta > 0$, if $p \geq C(k,\delta) \cdot n^{-\frac{1}{k-1}}$, then $[n]_p$ is w.h.p. (δ, k)-Szemerédi.

Sparse random analogues

Metatheorem (Conlon, Gowers; Samotij)

'dense' stability result
$+$
removal lemma
$\Longrightarrow$ 'sparse' stability result

Theorem (Conlon, Gowers)

For every $k \geq 2$ and every $\delta > 0$, there exist C and $\varepsilon > 0$ such that if $p \geq Cn^{-\frac{2}{k+2}}$, then a.a.s. every K_{k+1}-free subgraph of $G(n,p)$ with at least $(1 - \frac{1}{k} - \varepsilon)\binom{n}{2}p$ edges may be made k-partite by removing at most $\delta n^2 p$ edges.

Sparse analogues of counting problems

Definition

For a hypergraph $\mathcal{H}$, let

$$\mathcal{I}(\mathcal{H}) := \text{ independent sets in } \mathcal{H}.$$

Definition

Let V be a (finite) set. A family $\mathcal{F} \subseteq \mathcal{P}(V)$ is increasing (an upset) if $A \in \mathcal{F}$ and $B \supseteq A$ imply $B \in \mathcal{F}$.

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Let V be a (finite) set. A family $\mathcal{F} \subseteq \mathcal{P}(V)$ is increasing (an upset) if $A \in \mathcal{F}$ and $B \supseteq A$ imply $B \in \mathcal{F}$.

Definition

Let $\mathcal{H}$ be a k-uniform hypergraph, $\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$ an upset, and $\varepsilon > 0$. We say that $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-dense if for every $A \in \mathcal{F}$,

$$e(\mathcal{H}[A]) \geq \varepsilon e(\mathcal{H}).$$

Refined framework

Example

The following hypergraph is $(\mathcal{F}, \varepsilon)$-dense:

- $V(\mathcal{H}) = [n]$,
- $E(\mathcal{H}) =$ k-term APs,
- $\mathcal{F} = \{A \subseteq [n] \colon |A| \geq \delta n\}$.

Refined framework

Example

The following hypergraph is $(\mathcal{F}, \varepsilon)$-dense:

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Example

The following hypergraph is also $(\mathcal{F}, \varepsilon)$-dense:

- $V(\mathcal{H}) = E(K_n)$,
- $E(\mathcal{H}) =$ edge sets of copies of K_{k+1},
- $\mathcal{F} =$ graphs with at least $(1 - 1/k + \varepsilon)\binom{n}{2}$ edges.

Main result

Theorem (Balogh, Morris, Samotij)

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For every k and ε, there is $m_0 = m_0(N)$ such that if $\mathcal{H}$ is an N-vertex k-uniform hypergraph which

- *is $(\mathcal{F}, \varepsilon)$-dense for some upset $\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$ and*
- *satisfies certain technical conditions, [bounds on degrees, co-degrees]*

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- *satisfies certain technical conditions, [bounds on degrees, co-degrees]*

then there are

- *a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leq m_0}$ and*
- *functions $f : \mathcal{S} \to \mathcal{F}^c$ and $g : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$*

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- *functions $f : \mathcal{S} \to \mathcal{F}^c$ and $g : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$*

such that for every $I \in \mathcal{I}(\mathcal{H})$

$$g(I) \subseteq I \quad \text{and} \quad I \setminus g(I) \subseteq f(g(I)).$$

Corollaries

Implies most results of Conlon-Gowers, and Schacht:

- Sparse Szemerédi Theorem.
- Sparse Turán Theorem.
- Sparse Erdős- Stone Theorem (for balanced graphs).
- Sparse stability theorem.

Proof is much shorter and simpler!

Corollaries – Turán problem

Theorem (Balogh, Morris, Samotij)

For every k and $\delta > 0$, if $m \geq C(k, \delta)n^{2-2/(k+2)}$, then almost every K_{k+1}-free n-vertex graph with m edges can be made k-partite by removing from it at most δm edges.

Corollaries – Szemerédi Theorem

Theorem (Balogh, Morris, Samotij)

For every $k \geq 3$ and $\delta > 0$, if $m \geq C(k,\delta)n^{1-\frac{1}{k-1}}$, then

$$\#m\text{-subsets of } [n] \text{ with no } k\text{-term AP} \leq \binom{\delta n}{m}.$$

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Corollaries – Szemerédi Theorem

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Proof.

$m := \delta' pn, \delta := \delta'/e^2$.

$P\big([n]_p$ contains a k-term AP-free set of size $\delta' np\big)$

$$\leq \binom{\delta n}{m} \cdot p^m \leq \left(\frac{e\delta np}{\delta' np}\right)^{\delta' np} = o(1).$$

$\square$

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On the structure of subsets of $[n]$ with no k-term AP:

Theorem (Balogh, Morris, Samotij)

For every $k \geq 3, \delta > 0$ there is a C such that for

$$t = 2^{Cn^{1-1/k} \log n}.$$

there are $F_1, \ldots, F_t \subset [n]$,
each of size at most δn,
*such that for **every** subset of $[n]$ with no k-term AP there is an F_i containing it.*

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- Conlon, Gowers, Samotij, Schacht (2012++)
  Counterexamples even for the counting lemma are rare in random graphs!

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- density conditions imply that always many vertices are removed, i.e. $|f(S)|$ is small.