First Fit and Best Fit bin packing: A new analysis

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A Brief Review of Bin Packing

Bin packing

- Input: Sequence of **items** $a_1, \ldots, a_n \in [0, 1]$.
- Output: Assign into **bins** of size 1.
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Complexity results

- It is NP-hard to decide if $OPT(I) = 2$.
  Thus it is NP-hard to approximate with ratio $< 3/2$.
- There exists an asymptotic approximation scheme.
  I.e., in polynomial time we can pack the items into $(1 + \varepsilon)OPT(I) + 1$ bins.

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Asymptotic approximation ratio

There exists a constant C such that for each instance I, the algorithm gives

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Two online algorithms

First Fit

Packs items one by one, always into the **first** bin where it fits.
Opens a new bin only when necessary.

Best Fit

Packs items one by one, always into the **most full** bin where it fits.
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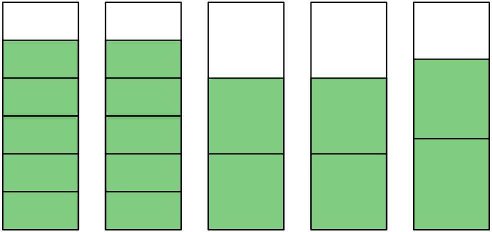

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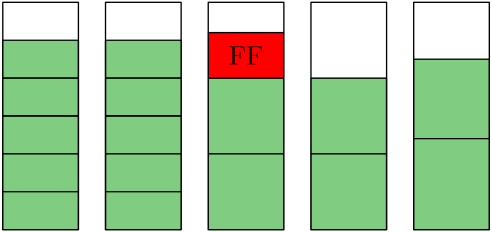

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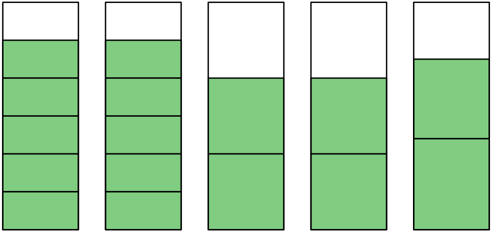

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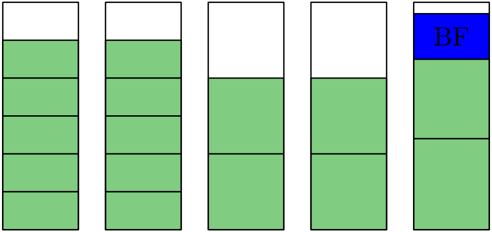

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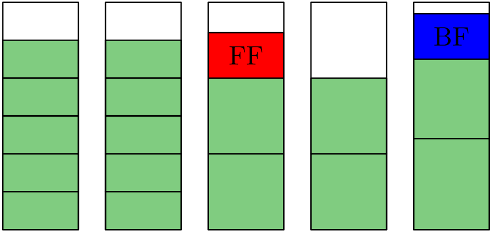
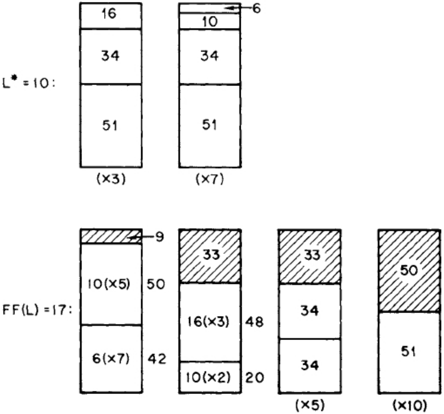


Known Results

The asymptotic approximation ratio of both First Fit and Best Fit is equal to 1.7. More precisely, $FF, BF \leq \lceil 1.7 \cdot OPT \rceil$.
Example with $FF = BF = 17$ and $OPT = 10$:



Bin size is 101.

Overview of this talk

1. An easy proof of the 1.7 asymptotic approximation ratio for First Fit.
2. An extension for Best Fit.
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Main Technique

Classical technigue: **Weight functions**.
Find a weight of items such that

- Each bin in OPT has weight ≤ 1.7.
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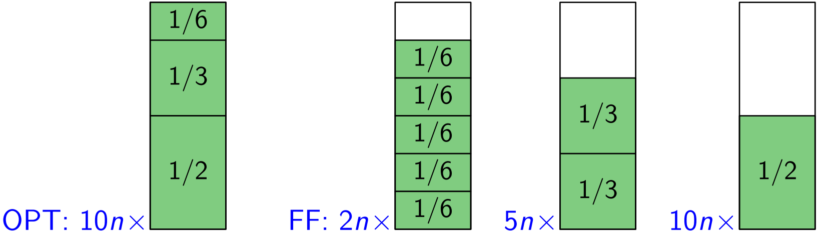
Combine weight functions with **amortized analysis**.

Idealized example for First Fit

Assume that the algorithm cannot have bins of size exactly 1.



Essentially, this can be achieved by changing the item sizes by a small amount.

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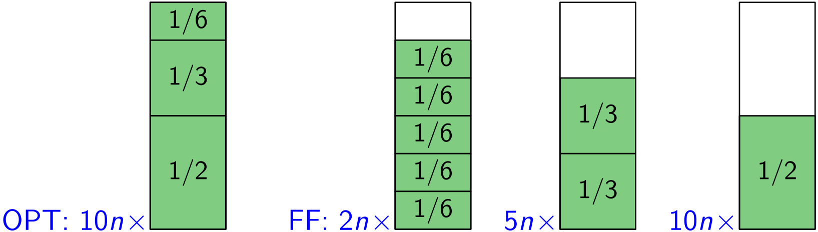

What should be the weights?

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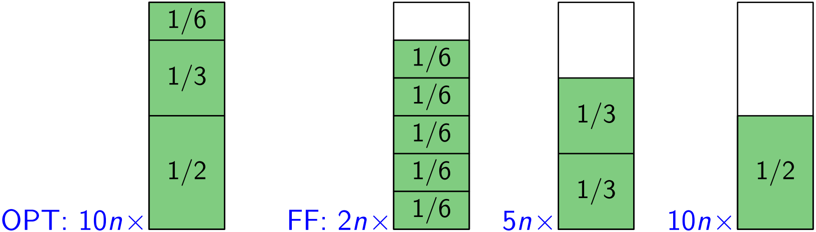

$w(1/6) = 0.2 \quad w(1/3) = 0.5 \quad w(1/2) = 1$

The weight function

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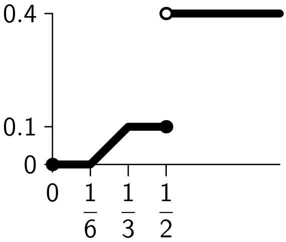
- $b(a) =$

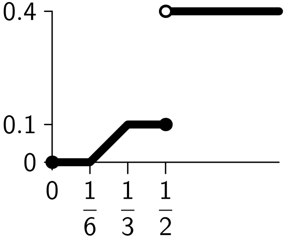
$$\begin{cases} 0 & \text{if } a \leq \frac{1}{6} \\ \frac{3}{5}\left(a - \frac{1}{6}\right) & \text{if } a \in [\frac{1}{6}, \frac{1}{3}] \\ 0.1 & \text{if } a \in [\frac{1}{3}, \frac{1}{2}) \\ 0.4 & \text{if } a > \frac{1}{2} \end{cases}$$



Offline bins

Each bin (a set of items of size ≤ 1) contains **bonus items**:

- either no item of size $> 1/2$ and at most 5 items with bonus at most 0.1 each (actually the total is < 0.3),
- or one item of size $> 1/2$ and at most 2 items with bonus at most 0.1 total.



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- the **total weight** is at most 1.7.

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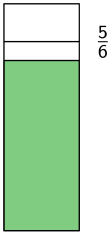
- Bins of size $\geq 5/6$ have weight ≥ 1.
- Bins with an item of size $> 1/2$ have weight ≥ 1.
- For the remaining bins with sizes in $(2/3, 5/6)$ we use **amortization**.
  We show that the scaled size of a bin plus the bonus of the **following** such bin is ≥ 1.

Amortization

For each bins with size in $(2/3, 5/6)$, at least two items, and no item $> 1/2$ we show that the scaled size of this bin plus the bonus of the **following** such bin is ≥ 1.



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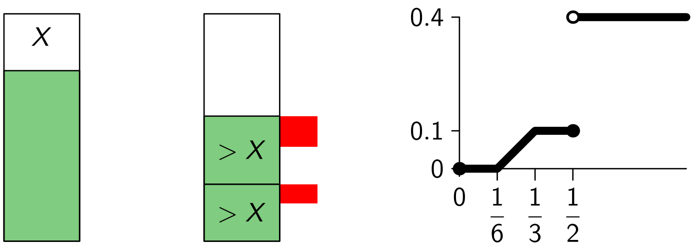

The result

$$FF(I) - 3 \leq w(I) \leq 1.7 \cdot OPT(I)$$

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Theorem

First Fit has **asymptotic** approximation ratio 1.7.

$$FF(I) \leq 1.7 \cdot OPT(I) + 3$$

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- The first item does not fit into any previous bin.
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- The first item does not fit into any previous bin.
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- We **close bins one by one**, paying 1 at each step.
  We always **close the largest bin**.
- The **bonus of one of the open bins** may be used to pay for the cost of closing **the previously closed bins**.

Amortization for Best Fit

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One of the open bins is the **boss**.
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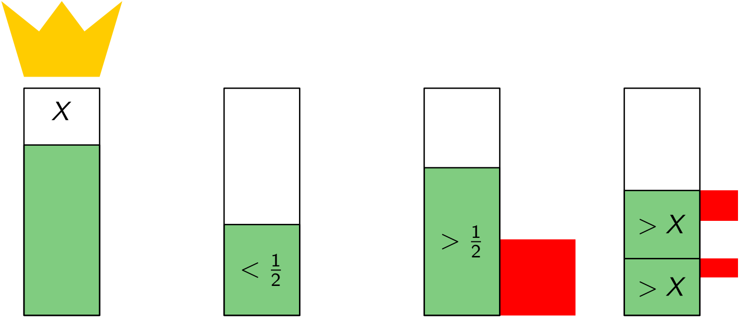
When the boss is closed

- We choose the new boss and
- we use also **its bonus** to pay the cost.

Choice of the new boss

The new boss is

- the oldest bin, unless it has a single item and its size is $\leq 1/2$;
- the second oldest bin otherwise.



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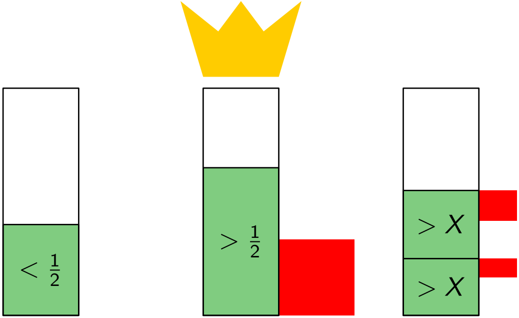

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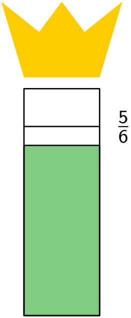

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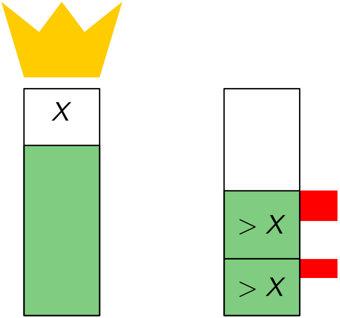

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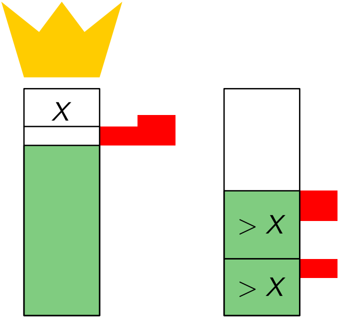

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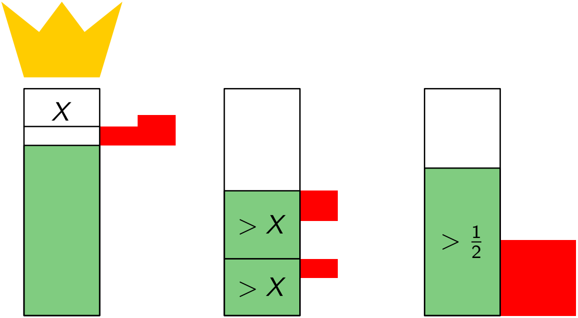

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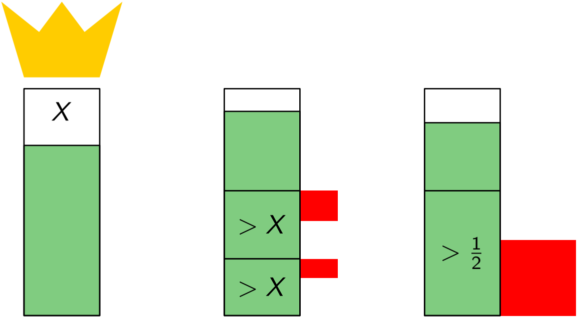
- Typically, the new boss has two items of size $> X$, which then have exactly the sufficient bonus.
- Or, even better, the new boss has an item of size $> \frac{1}{2}$.

A regular bin is closed



- If a regular bin has enough to pay for the boss, then it has enough for itself.

Results for Best Fit

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k-bounded-space Best Fit

- The most full bin is closed when k bins are open and the next item does not fit into any of them.
- For $k \geq 2$, k-**bounded-space Best Fit** has **asymptotic** approximation ratio 1.7.

Modified weights

Classify the First Fit bins

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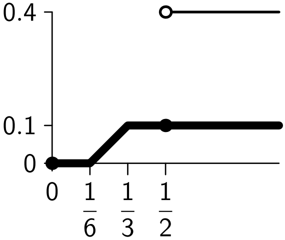
- Dedicated bins: **A single item**. Bonus 0.4, weight > 1. (Actually, 1 bin may have an item $\leq 1/2$, needs to have a smaller bonus. Still average weight > 1.)
- **Common bins:** The rest. Bonus as before, but only 0.1 for items $> 1/2$. We need to prove that these C bins have **total weight** $\geq C - 0.2$.



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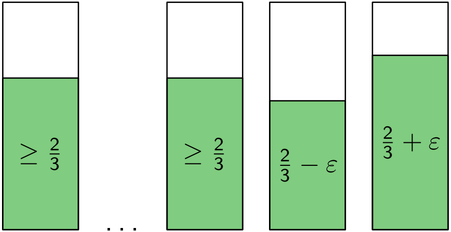

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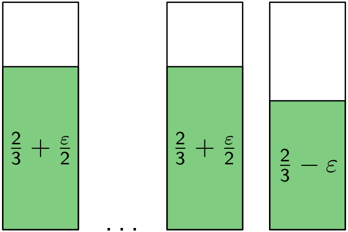
- The last common bin has size $< 2/3$: Harder, but works for large OPT. Some cases solved separately.



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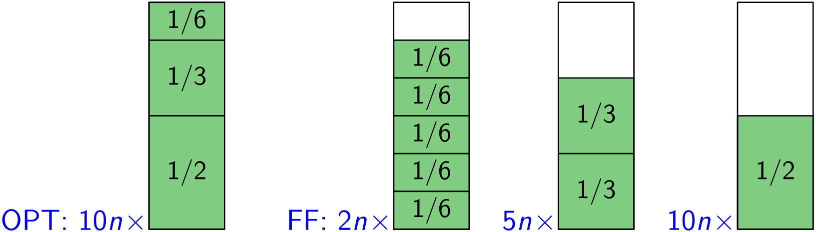
  - the weight of this bin is at most 1.6 and thus $w(I) \leq 1.7 \cdot OPT(I) - 0.1$;
  - the analysis for the common bins still holds, since the items with removed bonus are in a bin with two more items.

Idealized example revisited

Assume that the algorithm cannot have bins of size exactly 1.



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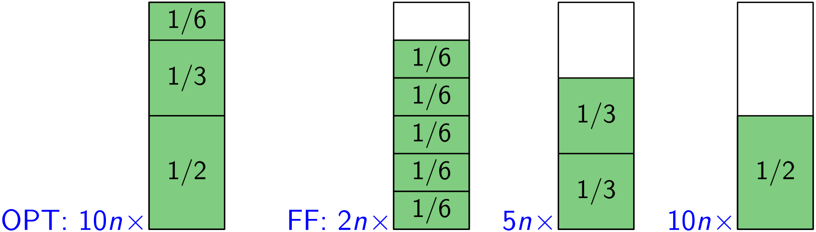

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We use this as a black box to get tight bounds.

Lower bounds

Suppose that $OPT = 10k + i$, $i = 0, \ldots, 9$.
Then the lower and upper bounds for First Fit are:

$i =$	0	1	2	3	4	5	6	7	8	9
$FF \geq 17k +$	-1	1	3	4	6	8	10	11	13	15
$FF \leq \lfloor 17k + 1.7i \rfloor = 17k +$	0	1	3	5	6	8	10	11	13	15

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