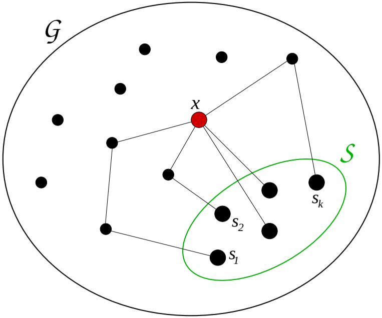
Resolving sets in finite projective planes

Marcella Takáts
(Eötvös Loránd University, Budapest)
Joint work with Tamás Héger

Finite Geometry Conference and Workshop
10-14 June 2013
Szeged, Hungary

Let $G = (V, E)$ be a simple graph.
$d(x, y)$: distance of x and y, $x, y \in V$
$S = \{s_1, s_2, \dots, s_k\}$ vertex set



$$d(x, s_1)$$
$$d(x, s_2)$$
$$\dots$$
$$d(x, s_k)$$

x is resolved by S if its distance list is different from all the other distance lists

Definition (Resolving set)

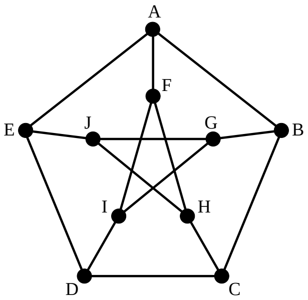
The subset $S = \{s_1, \ldots, s_k\} \subset V$ is a resolving set, if the ordered distance lists $(d(x, s_1), \ldots, d(x, s_k))$ are different for all $x \in V$.

In other words:
$S = \{s_1, \ldots, s_k\} \subset V$ is a resolving set $\iff$
$\forall x, y \in V \;\; \exists z \in S\colon d(x, z) \neq d(y, z)$.

Definition (Metric dimension)

The metric dimension $\mu(G)$ is the size of the smallest resolving set.

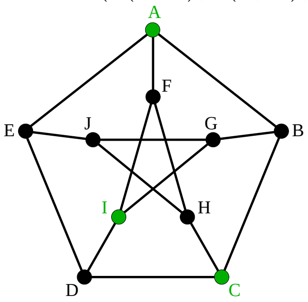
Example: Petersen graph



Example: Petersen graph

Resolving set: $A,\ C,\ I$

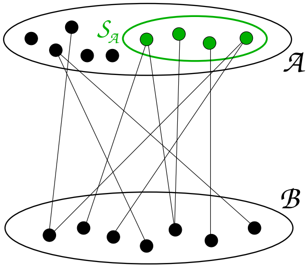
$x \in V$: $(d(x,A), d(x,C), d(x,I))$



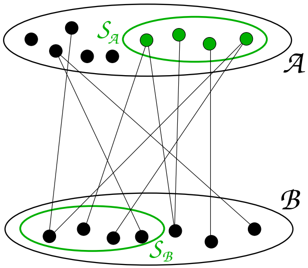
$A:$	$(0,2,2)$	$F:$	$(1,2,1)$
$B:$	$(1,1,2)$	$G:$	$(2,2,1)$
$C:$	$(2,0,2)$	$H:$	$(2,1,2)$
$D:$	$(2,1,1)$	$I:$	$(2,2,0)$
$E:$	$(1,2,2)$	$J:$	$(2,2,2)$

Metric dimension: $\mu(Petersen) = 3$

Let $G = (A \cup B, E)$ be a bipartite graph.
split resolving set:

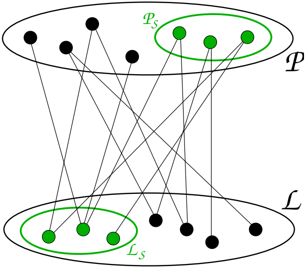


Let $G = (A \cup B, E)$ be a bipartite graph.
split resolving set:
$S_A \subset A$ resolves B
$S_B \subset B$ resolves A



S_A and S_B are called semi-resolving sets.
Next talk: Tamás Héger - semi-resolving sets!

Resolving sets in incidence graphs of finite projective planes



(P, ℓ) edge $\Leftrightarrow P \in \ell$

$G = (\mathcal{P}, \mathcal{L}, E)$: incidence graph of a finite projective plane

$S = (\mathcal{P}_S \cup \mathcal{L}_S)$ resolving set in $\Pi_q \iff$

S is a resolving set in the incidence graph

$d(P_1, P_2) = 2$, $d(\ell_1, \ell_2) = 2$, $d(P, \ell) = 1$ or 3

Motivation

metric dimension $\mu(G)$:

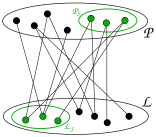
- first introduced by Harary and Melter and (independently) by Slater in the 1970s
- a survey of Bailey and Cameron (2011)
- connection with other graph parameters
- talk of Robert Bailey on the 23rd BCC, Exeter, July 2011:
- he asked the metric dimension of $G = (\mathcal{P}, \mathcal{L}, E)$
- construction of a resolving set of size $4q - 1$ (Bill Martin)

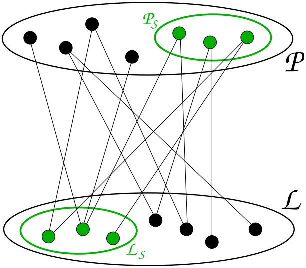
Motivation

- connection with other graph parameters: dimension jump
- $B \subset V(G)$ is a base: the only automorphism of G that fixes B pointwise is the identity
- $b(G)$ base size: the size of the smallest base of G
- $b(G) \leq \mu(G)$
- dimension jump: $\delta(G) = \mu(G) - b(G)$
- projective planes: highly symmetric graphs with large dimension jump
- Γ: incidence graph of $\mathrm{PG}(2, q)$,
  order of Γ: $n = 2(q^2 + q + 1)$
- $b(\Gamma) \leq 5$
- $\delta(\Gamma) \geq 4q - 9 \sim 2\sqrt{2n}$

Notation

- Π_q: projective plane of order q
- $S = \mathcal{P}_S \cup \mathcal{L}_S$: resolving set in the incidence graph of Π_q
- PQ: line joining two distinct points P and Q
- $[P]$: set of lines through a point P
- $[\ell]$: set of points on a line ℓ
- tangent line $\longleftrightarrow$ 1-covered point
- skew line $\longleftrightarrow$ not covered point





(P, ℓ) edge $\Leftrightarrow P \in \ell$

$G = (\mathcal{P}, \mathcal{L}, E)$: incidence graph of a finite projective plane
$S = (\mathcal{P}_S \cup \mathcal{L}_S)$ resolving set in $\Pi_q \iff$
$\forall \ell \in \mathcal{L}$: $\ell \in \mathcal{L}_S$, or $\ell \cap \mathcal{P}_S$ is unique
$\forall P \in \mathcal{P}$: $P \in \mathcal{P}_S$, or $[P] \cap \mathcal{L}_S$ is unique

Lemma

Let $S = \mathcal{P}_S \cup \mathcal{L}_S$, ℓ be a line in Π_q. If $|[\ell] \cap \mathcal{P}_S| \geq 2$ then ℓ is resolved by S.
Dually, let P be a point in Π_q. If $|[P] \cap \mathcal{L}_S| \geq 2$ then P is resolved by S.

- Points and lines in S are resolved
- (At least 2-)secants are resolved
- (At least) 2-covered points are resolved

We have to distinguish:

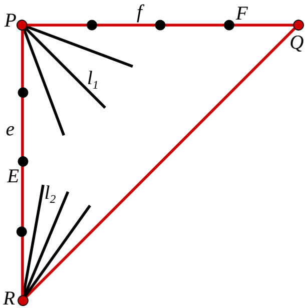
- tangents and skew lines (to $\mathcal{P}_S$)
- 1-covered points and not covered points (by $\mathcal{L}_S$)

Proposition

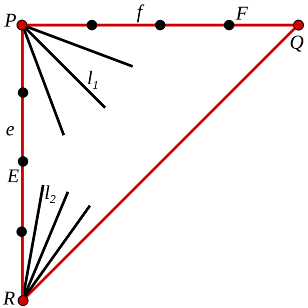
$S = \mathcal{P}_S \cup \mathcal{L}_S$ is a resolving set in a finite projective plane if and only if the following properties hold for S:

- *P1 There is at most one outer line skew to $\mathcal{P}_S$.*
- *P1' There is at most one outer point not covered by $\mathcal{L}_S$.*
- *P2 Through every inner point there is at most one outer line tangent to $\mathcal{P}_S$.*
- *P2' On every inner line there is at most one outer point that is 1-covered by $\mathcal{L}_S$.*

Example

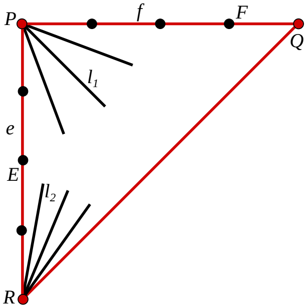


- P1 Outer line skew to $\mathcal{P}_S$: RQ.
- P1' Outer point not covered by $\mathcal{L}_S$: Q.
- P2 Through $F \in (f \cap \mathcal{P}_S)$ there is no outer tangent line. Through $E \in (e \cap \mathcal{P}_S)$ outer tangent line: EQ.
- P2' On $\ell_1 \in ([P] \cap \mathcal{L}_S)$ outer 1-covered point: $\ell_1 \cap RQ$. On $\ell_2 \in ([R] \cap \mathcal{L}_S)$ there is no outer 1-covered point.



- P1 Outer line skew to $\mathcal{P}_S$: RQ.
- P1' Outer point not covered by $\mathcal{L}_S$: Q.
- P2 Through $F \in (f \cap \mathcal{P}_S)$ there is no outer tangent line. Through $E \in (e \cap \mathcal{P}_S)$ outer tangent line: EQ.
- P2' On $\ell_1 \in ([P] \cap \mathcal{L}_S)$ outer 1-covered point: $\ell_1 \cap RQ$ On $\ell_2 \in ([R] \cap \mathcal{L}_S)$ there is no outer 1-covered point.

Example



$$|S| = |\mathcal{P}_S| + |\mathcal{L}_S| = 2q - 2 + 2q - 2 = 4q - 4$$

The main result

Proposition (T. Héger, M. Takáts)

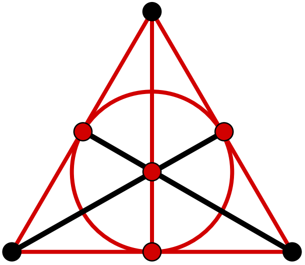
The metric dimension of a projective plane of order $q \geq 23$ is $4q - 4$.

- Classification of the resolving sets of size $4q - 4$, if $q \geq 23$.

If q is small ($q < 23$):

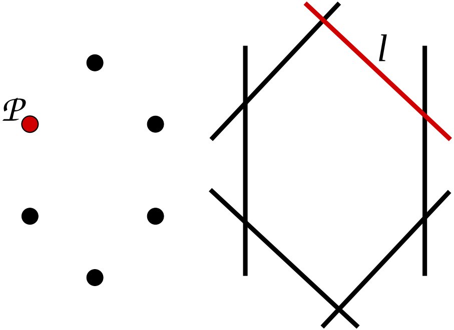
Fano plane:

$\mu(\mathrm{PG}(2,2)) = 5$



$\mu(\mathrm{PG}(2,4)) = 10$

construction: hyperoval



Sketch of the proof

- Suppose $S = \mathcal{P}_S \cup \mathcal{L}_S$ is a resolving set, $|S| \leq 4q - 4$.
- Show that $\mu(\Pi_q) = 4q - 4$.
- Characterize the resolving sets of size $4q - 4$.
- combinatorial methods
- we use duality

Sketch of the proof

Proposition

$2q - 5 \leq |\mathcal{P}_S| \leq 2q + 1$, $2q - 5 \leq |\mathcal{L}_S| \leq 2q + 1$.

Proof

t: *the number of tangents not in* $\mathcal{L}_S$
Bounds: $t \leq |\mathcal{P}_S|$ *and* $|\mathcal{P}_S| + |\mathcal{L}_S| \leq 4q - 4$
Double counting the pairs:
$\star = \{(P, \ell)\colon P \in \mathcal{P}_S, P \in [\ell], |[\ell] \cap \mathcal{P}_S| \geq 2\}$
$2(q^2 + q + 1 - 1 - t - |\mathcal{L}_S|) \leq \star \leq |\mathcal{P}_S|(q + 1) - t$
We get $2q - 5 \leq |\mathcal{P}_S|$ *and dually* $2q - 5 \leq |\mathcal{P}_S|$. □

Sketch of the proof

Proposition

Let $q \geq 89$. Then any line intersects $\mathcal{P}_S$ in either ≤ 7 or $\geq q-4$ points.

Proof

ℓ: *secant of* S, $|[\ell] \cap \mathcal{P}_S| = x$, $2 \leq x \leq q$

Let $P \in [\ell] \setminus \mathcal{P}_S$.

$s(P), t(P)$: *number of skew/tangent lines to* $\mathcal{P}_S$ *through* P, *resp.*

s: *the number of skew lines*

t: *the number of tangents intersecting* $\ell \setminus \mathcal{P}_S$

Bounds: $s \leq |\mathcal{L}_S| + 1$ *and* $s + t \leq |\mathcal{L}_S| + (1 + |\mathcal{P}_S| - x)$

$\Longrightarrow 2s + t \leq 2|\mathcal{L}_S| + |\mathcal{P}_S| - x + 2$;

$|\mathcal{P}_S| + |\mathcal{L}_S| \leq 4q - 4$

Proof (cont'd)

Counting the points of $\mathcal{P}_S$ *on* ℓ *and on* $[P]$:
$2q+1 \geq |\mathcal{P}_S| \geq x + t(P) + 2(q - t(P) - s(P))$
$x \leq 2s(P) + t(P) + 1$
Adding up the inequalities $\forall P \in [\ell] \setminus \mathcal{P}_S$:
$(q+1-x)x \leq 2s + t + (q+1-x)$.
We get $x^2 - (q+3)x + 7q \geq 0 \Rightarrow$
$x \leq 7$ *or* $x \geq q-4$, *if* $q \geq 89$. □

Sketch of the proof

Proposition

Let $q \geq 47$. Then there exist two lines intersecting $\mathcal{P}_S$ in at least $q-4$ points.

Proof

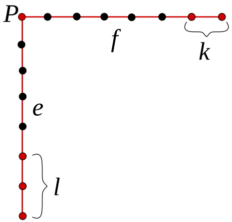
Suppose to the contrary: there is ≤ 1 line intersecting $\mathcal{P}_S$ in at least $q-4$ points.
ℓ*: the longest secant, let $x = |[\ell] \cap \mathcal{P}_S| \geq 2$.*
Note that $x \leq 7$ is also possible.
n_i*: the number of i-secants different from ℓ*
Let $n_0 = s$, $n_1 = t$, and let $b = |\mathcal{P}_S|$
The standard equations yield:
$\sum_{i=2}^{7} n_i = q^2 + q + 1 - s - t - 1$
$\sum_{i=2}^{7} i n_i = (q+1)b - t - x$
$\sum_{i=2}^{7} i(i-1)n_i = b(b-1) - x(x-1)$

Proof (cont'd)

$0 \leq \sum_{i=2}^{7}(i-2)(7-i)n_i =$
$= -\sum_{i=2}^{7} i(i-1)n_i + 8\sum_{i=2}^{7} in_i - 14\sum_{i=2}^{7} n_i =$
$= -b^2 + (8q+9)b + x(x-9) + 6(s+t) + 8s - 14(q^2+q)$.
Bounds: $s + t \leq |\mathcal{P}_S| + |\mathcal{L}_S| + 1 \leq 4q-3$, $s \leq |\mathcal{L}_S| + 1 \leq 2q+2$
and $x \leq q+1$
we get $0 \leq -b^2 + (8q+9)b - 13q^2 + 19q - 10$.
$b = |\mathcal{P}_S| \leq 2q+1$
For $b = 2q+2$*:*
$-q^2 + 45q - 20 < 0$ *if* $q \geq 47$.
Hence $b > 2q+2$*, a contradiction.* □

Sketch of the proof

There exist two distinct lines e, f: $e \cap f = P$
$|[e] \cap (\mathcal{P}_S \setminus P)| = q - l$ and $|[f] \cap (\mathcal{P}_S \setminus P)| = q - k$ with $k \leq l \leq 5$.



Proposition

Suppose $q \geq 23$. Then $k + l \leq 3$. Moreover, $l = 3$ is not possible.

Dually:

Proposition

There exist two points P, R such that $|[P] \cap \mathcal{L}_S|$ and $|[R] \cap \mathcal{L}_S|$ is at least $q - 4$.

For the points P, R:
$|[P] \cap \mathcal{L}_S| = q - l'$ and $|[R] \cap \mathcal{L}_S| = q - k'$ with $k', l' \leq 5$.

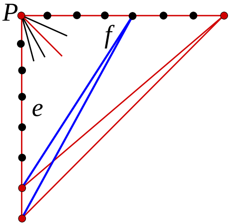
Proposition

$k' + l' \leq 3$. *Moreover,* $k' = 3$ *or* $l' = 3$ *is not possible.*

Suppose: $|\mathcal{P}_S| \leq |\mathcal{L}_S|$.
$\implies |\mathcal{P}_S| \leq 2q - 2$.

Sketch of the proof

- $k + l \leq 3 \Longrightarrow |\mathcal{P}_S| \geq 2q - 3$
- $|S| = |\mathcal{P}_S| + |\mathcal{L}_S| \leq 4q - 4$, we assumed $|\mathcal{P}_S| \leq |\mathcal{L}_S|$
- If $|\mathcal{P}_S| = 2q - 3$ then $|\mathcal{L}_S| \geq 2q - 1$

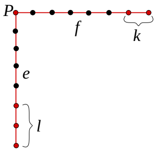


- If $|\mathcal{P}_S| = 2q - 2$ then $|\mathcal{L}_S| \geq 2q - 2$

Corollary

Suppose $q \geq 23$. The metric dimension of Π_q is $\mu = 4q - 4$.

Sketch of the proof



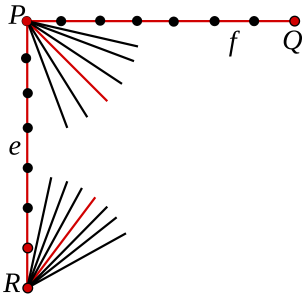
Proposition

There exists a point $R \in e \setminus P$ *such that* $|([R] \setminus e) \cap \mathcal{L}_S| \geq q - 1$*. Moreover, if* $l = 2$ *then* $R \notin \mathcal{P}_S$*.*

Proposition

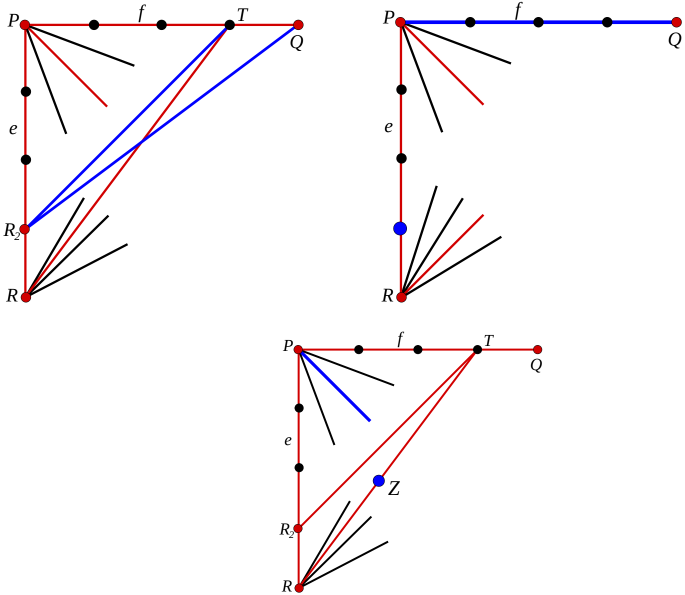
Let e*,* f *be the lines such that* $e \cap f = P$*,* $|[e] \cap (\mathcal{P}_S \setminus P)| = q - l$ *and* $|[f] \cap (\mathcal{P}_S \setminus P)| = q - k$*,* $k + l \leq 3$*,* $l \neq 3$*. Then* $|([P] \setminus \{e, f\}) \cap \mathcal{L}_S| \geq q - 2$*.*

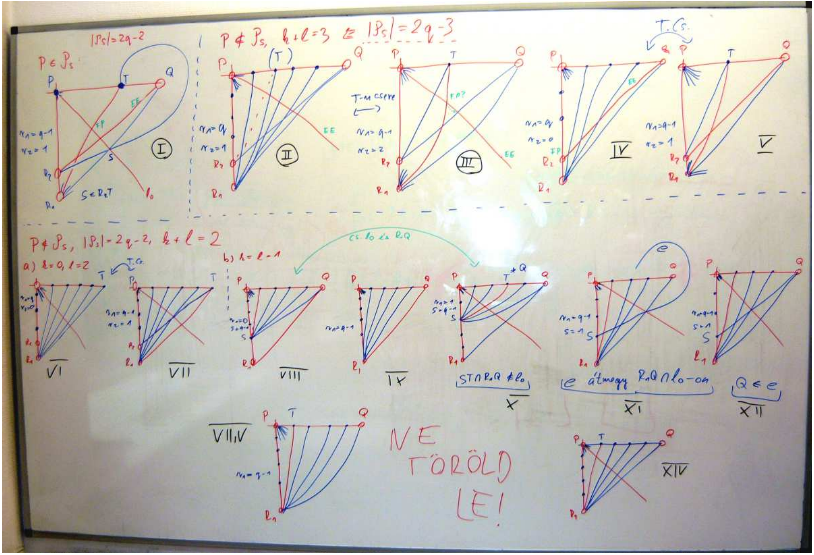
So we get the following:

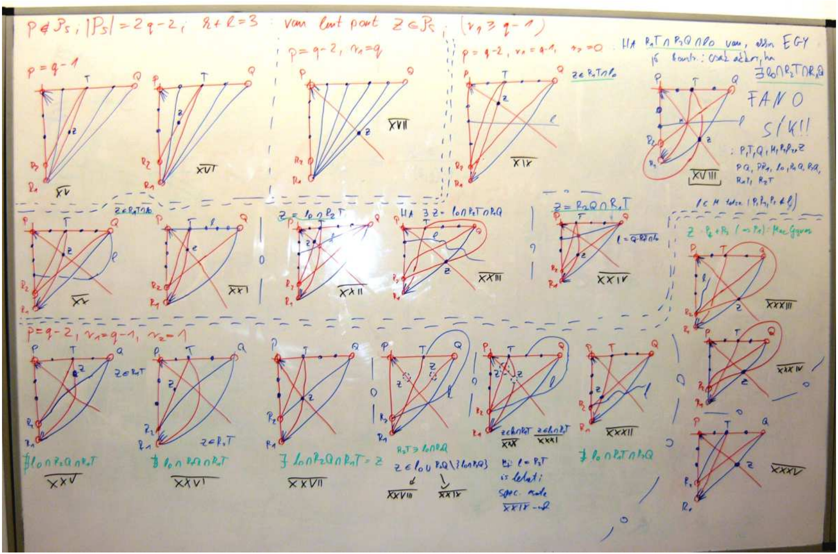


And we need 2 more objects in addition:
2 lines or 1 point and 1 line
surprisingly many (more than 30) different types

Some constructions




|P_S|=2q-2
P ∈ P_S
P ∉ P_S, k+l=3 ⇐ |P_S|=2q-3
T-m csere
T.Cs.
P ∉ P_S, |P_S|=2q-2, k+l=2
a) k=0, l=2
b) k=l=1
I
II
III
IV
V
VI
VII
VIII
IX
X
XI
XII
XIV
VII,V
NE TÖRÖLD LE!


FANO
SÍK!!

Thanks for your attention!