Translation ovoids in finite classical polar spaces

Alessandro Siciliano

Department of Mathematics, Computer Science and Economy

University of Basilicata

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Polarities of vector spaces

Let $V = V(n, K)$ be an $n-$dimensional $K-$vector space, K a field.

Let $\sigma : K \to K$ be an automorphism.

A *$\sigma-$sesquilinear form* on V is a map $B : V \times V \to K$ such that

$$\begin{aligned} B(u + v, w) &= B(u, w) + B(v, w) \\ B(u, v + w) &= B(u, v) + B(u, w) \\ B(au, bv) &= a\,B(u, v)b^{\sigma} \end{aligned}$$

for all $u, v, w \in V$, all $a, b \in K$.

If $\sigma = 1$, the form is said to be *linear*.

B is called *non-degenerate* if, $B(u, v) = 0$ for all $v \in V$ implies $u = 0$, and, $B(u, v) = 0$ for all $u \in V$ implies $v = 0$.

A sesquilinear form B such that $B(u, v) = 0$ implies $B(v, u) = 0$ for all $u, v \in V$ is called *reflexive*.

A reflexive $\sigma-$sesquilinear form B is called:

- Alternating: if $\sigma = 1$ and $B(v, v) = 0$ for all $v \in V$
- Symmetric: if $\sigma = 1$ and $B(u, v) = B(v, u)$ for all $u, v \in V$
- Hermitian: if $\sigma^2 = 1, \sigma \neq 1$ and $B(u, v) = B(v, u)^\sigma$ for all $u, v \in V$

Note that if B is alternating then $B(v, u) = -B(u, v)$, i.e. in general B is antisymmetric.

If $\mathrm{char}\, K \neq 2$ then the concepts of alternating form and antisymmetric form are equivalent.

If $\mathrm{char}\, K = 2$, each alternating form is also symmetric.

Quadratic forms

If $\mathrm{char}\, K \neq 2$ and B is a symmetric form the map

$$\begin{array}{rccc} Q: & V & \longrightarrow & K \\ & v & \longmapsto & B(v,v) \end{array}$$

satisfies

$$\begin{aligned} Q(av) &= a^2 Q(v) \\ Q(u+v) &= Q(u) + 2B(u,v) + Q(v). \end{aligned}$$

Q is called the *quadratic form* associated with B.

Also $B(u,v) = \frac{1}{2}[Q(u+v) - Q(u) - Q(v)]$ is uniquely determined by Q.

If $\mathrm{char}\, K = 2$, the above does not apply.

We define a *quadratic form* to be a function $Q : V \to K$ such that

$$Q(av) = a^2 Q(v)$$

and

$$B(u, v) = Q(u + v) - Q(u) - Q(v)$$

is a bilinear form.

Then B is uniquely determined by Q and is called *the polar form* of Q.

If $\mathrm{char}\, K = 2$,

$$B(u, u) = Q(u + u) + Q(u) + Q(u) = 0$$

for all $u \in V$, so B must be alternating and, since $\mathrm{char}\, K = 2$, B is also symmetric.

Moreover:

- Q is not uniquely determined by B.
- there are symmetric forms that are not the polar form of any quadratic form.

In our considerations, we assume that when B is symmetric it arises as the polar form of a quadratic form.

Let Q be a quadratic form on V with polar form B. Then Q is called *non-degenerate* if $B(u, v) = 0 = Q(u)$ for all $v \in V$ implies $u = 0$.

The space (V, B) is called *symplectic*, *orthogonal* or *unitary geometry* according to whether B is a non-degenerate alternating, symmetric or hermitian form on V.

Polarities of projective spaces

A pair of vectors (u, v) is called *orthogonal* if $B(u, v) = 0$.

For any subspace X of V the set

$$X^{\perp} := \{v \in V : B(u, v) = 0 \text{ for all } u \in X\}$$

is called the *orthogonal complement* of X.

The *projective geometry* $\mathbb{P}(V)$ is the set of all subspaces of V ordered by set inclusion.

A *polarity* of $\mathbb{P}(V)$ is a correlation π of order 2 and the pair $(\mathbb{P}(V), \pi)$ is called a *polar geometry*.

Any non-degenerate sesquilinear form on V defines a polarity of $\mathbb{P}(V)$:

$$\begin{array}{rccc} \perp: & \mathbb{P}(V) & \to & \mathbb{P}(V) \\ & \langle u \rangle & \mapsto & \langle u \rangle^{\perp} \end{array}$$

The space $(\mathbb{P}(V), \perp)$ is called projective *symplectic*, *orthogonal* or *unitary geometry* according to whether $\perp$ arises from an alternating, symmetric or hermitian non-degenerate bilinear form.

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Birkhoff-von Neumann Theorem

If $\dim V \geq 3$, every polarity of $\mathbb{P}(V)$ is symplectic, orthogonal or unitary.

Matrix of a bilinear form

If $\{v_1, \dots, v_n\}$ is a basis for V, $u = \sum_i a_i v_i$, $v = \sum_i b_i v_i$ and $\widehat{B} = B(v_i, v_j)$, then

$$B(u, v) = \sum_{i,j} a_i B(v_i, v_j) \bar{b_j} = \mathbf{a}^t \widehat{B} \mathbf{b}^\sigma.$$

where $\mathbf{a} = (a_1, \dots, a_n)^t$ and $\mathbf{b}^\sigma = ({b_1}^\sigma, \dots, {b_n}^\sigma)^t$.

Witt index of a sesquilinear form

Definitions:

- A non-zero vector u is *isotropic* if $B(u,u) = 0$.
- A subspace X of V is *totally isotropic* if $X \subseteq X^{\perp}$ i.e. $B(u,v) = 0$ for all $u, v \in X$.
- A non-zero vector u is *singular* if $Q(u) = 0$ and a subspace X is *totally singular* if $Q(u) = 0$ for all $u \in X$.
- A pair of vectors (u,v) such that u and v are isotropic and $B(u,v) = 1$ is called a *hyperbolic pair*.

A totally isotropic subspace is called *maximal* if it not properly contained in a totally isotropic subspace.

Theorem

Any two maximal totally isotropic subspaces of (V, B) have the same dimension, and every totally isotropic subspace is contained in one of maximal dimension.

This common dimension is called the *Witt index* of the sesquilinear form B.

The maximal totally isotropic subspaces are called generators of the polar space.

The set of all totally isotropic subspaces with respect to a non-degenerate sesquilinear form B on V, is called a *symplectic* , *orthogonal* or *unitary polar space* according to whether B is alternating, symmetric or hermitian.

Notation

$$\begin{array}{ll} B \textit{ alternating}: & \mathcal{W}(n-1,K) \\ B \textit{ symmetric}: & \mathcal{Q}(n-1,K) \\ B \textit{ hermitian}: & \mathcal{H}(n-1,K) \end{array}$$

The above polar spaces are called the *classical polar spaces*.

Finite classical polar spaces

Let $K = \mathrm{GF}(q)$, q a prime power.

$V = V(n, q)$ is a finite vector space over $\mathrm{GF}(q)$.

$\mathbb{P}(V) = \mathbb{P}(n, q)$ is a finite projective space over $\mathrm{GF}(q)$.

Then we have the *finite classical polar spaces*.

B a non-degenerate alternating form on V

Note that the points of $\mathcal{W}(n-1, K)$ are all the point of $\mathbb{P}(V)$.

We can always decompose V as

$$V = W_1 \bigoplus W_2 \bigoplus \cdots \bigoplus W_m,$$

that is a direct sum of mutually orthogonal subspaces where:

- W_i is a *hyperbolic* 2-space, i.e. $W_i = \langle v_i, w_i \rangle$, $B(v_i, v_i) = 0 = B(w_i, w_i)$, $B(v_i, w_i) = 1$

We see that the Witt index of (V, B) is m and we have just one symplectic geometry in $V = V(2m, q)$.

Finite unitary polar spaces

B a non-degenerate hermitian form on V

$V = V(n, q)$, $n \geq 2$, contains singular vectors and we can decompose V as

$$V = W_1 \bigoplus W_2 \bigoplus \cdots \bigoplus W_m \bigoplus W,$$

where:

- W_i is a *hyperbolic* 2-space
- W nonsingular and $\dim W \in \{0, 1\}$

We see that the Witt index of (V, B) is m and the hermitian geometry (V, B) is determined, up to isomorphisms, by m and W.

We have two different Hermitian polar spaces:

$W = 0$

$$\mathcal{H}(2m-1, q) : X_1 Y_1^q + \ldots + X_m Y_m^q = 0$$

$\dim W = 1$

$$\mathcal{H}(2m, q) : X_1 Y_1^q + \ldots + X_m Y_m^q + Z^{q+1} = 0$$

Finite orthogonal polar spaces

B a non-degenerate symmetric form on V

$V = V(n,q)$ contains singular vectors and we can decompose V as

$$V = W_1 \bigoplus W_2 \bigoplus \cdots \bigoplus W_m \bigoplus W,$$

where

- W_i is a *hyperbolic* 2-space
- W nonsingular and $\dim W \in \{0,1,2\}$

We see that the Witt index of V is m and the orthogonal geometry (V,B) is determined, up to isomorphisms, by m and W.

We have three different orthogonal polar spaces:

Hyperbolic: $W = 0$

$$\mathcal{Q}^+(2m-1, q) : X_1 Y_1 + \ldots + X_m Y_m = 0$$

Parabolic: $\dim W = 1$

$$\mathcal{Q}(2m, q) : X_1 Y_1 + \ldots + X_m Y_m + Z^2 = 0$$

Elliptic: $\dim W = 2$

$$\mathcal{Q}^-(2m+1) : X_1 Y_1 + \ldots + X_m Y_m + f(X, Y) = 0,$$

with $f(X, Y)$ an irreducible homogeneous quadratic polynomial over $\mathrm{GF}(q)$.

The finite classical polar spaces are

$$\mathcal{W}(2m-1,q)$$

$$\mathcal{Q}^{+}(2m-1,q),\ \mathcal{Q}(2m,q),\ \mathcal{Q}^{-}(2m+1,q)$$

$$\mathcal{H}(2m-1,q^2),\ \mathcal{H}(2m,q^2)$$

Abstract polar space

A *(abstract) polar space* of rank $m \geq 2$, consists of a set $\mathcal{P}$ of points, together with a set of subsets of $\mathcal{P}$, called subspaces, that satisfy certain axioms :

- (T1) Every subspace, together with its subspaces, is a projective space of dimension at most $m-1$.
- (T2) The intersection of any family of subspaces is a subspaces.
- (T3) If U is a subspace of dimension $m-1$ and P a point not in U, then the union of the lines joining P to points of U is a subspace of dimension $m-1$ and $U \cap W$ is a hyperplane in both U and W.
- (T4) There exist two disjoint subspaces of dimension $m-1$.

A polar space of rank two is called a *generalized quadrangle*.

Tits-Veldkamp Theorem

If $\mathcal{P}$ is finite and has rank ≥ 3, then $\mathcal{P}$ is classical (the rank being the Witt index).

Hence, a finite polar space is either classical (of rank ≥ 3) or a generalized quadrangle.

A *(finite) generalized quadrangle* GQ of order (s, t) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ *points* and $\mathcal{B}$ *lines* are disjoint (nonempty) sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x\mathcal{I}M\mathcal{I}y\mathcal{I}L$.

Ovoids of finite classical polar spaces

Let $\mathcal{P}$ denotes a finite classical polar space.

An *ovoid* of $\mathcal{P}$ is a set of points intersecting every generator in exactly one point.

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Ovoid numbers

Polar space	Ovoid number
$\mathcal{W}(2m-1,q)$	q^m+1
$\mathcal{H}(2m,q)$	$q^{2m+1}+1$
$\mathcal{H}(2m-1,q)$	$q^{2m-1}+1$
$\mathcal{Q}^-(2m+1,q)$	$q^{m+1}+1$
$\mathcal{Q}(2m,q)$	q^m+1
$\mathcal{Q}^+(2m-1,q)$	$q^{m-1}+1$

State of the art on existence and non-existence of ovoids

Symplectic polar spaces

$\mathcal{W}(3, q)$	q even: yes q odd: no
$\mathcal{W}(2m-1, q)$	$m \geq 3$: no

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Symplectic polar spaces

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Unitary polar spaces

$\mathcal{H}(2m,q^2)$	$m \geq 1$: no (Thas,1981)
$\mathcal{H}(3,q^2)$	yes
$\mathcal{H}(5,4)$	no (De Beule - Metsch, 2006)
$\mathcal{H}(2m-1,q^2)$	$q=p^h$, p prime, $p^{2m+1} > g(m,p)$: no

here $g(m,p) = \binom{2m+p}{2m-1}^2 - \binom{2m+p-2}{2m-1}^2$.

Orthogonal polar spaces

$\mathcal{Q}^-(2m+1,q)$	$m \geq 2$: no (Thas, 1981)
$\mathcal{Q}(4,q)$	yes
$\mathcal{Q}(6,q)$	q even: no $q = 3^h$: yes $q > 3$, q prime: no
$\mathcal{Q}(2m,q)$	q even, $m \geq 4$: no (Gunawardena - Moorhouse, 1997)
$\mathcal{Q}^+(3,q)$	yes
$\mathcal{Q}^+(5,q)$	yes
$\mathcal{Q}^+(7,q)$	$q = 2^h$: yes q odd prime: yes $q = 3^h$: yes $q \equiv 2 \bmod 3$: yes
$\mathcal{Q}^+(2m-1,2)$	$m \geq 5$:no (Kantor, 1982)
$\mathcal{Q}^+(2m-1,3)$	$m \geq 5$:no (Shult, 1989)
$\mathcal{Q}^+(2m-1,q)$	$q = p^h$, $p^{m-1} > \binom{2(m-1)+p}{p-1}$: no (Blokhuis - Moorhouse, 1995)

A. Klein, 2001

$H(2m-1,q^2)$, $q=p^h$, p prime, has no ovoids if $m>q^3-1$.

AS, 2008

$Q^+(2m-1,q)$, m odd, has no ovoids when $(m-1)/2>q^3+1$.

de Beule - Klein - Metsch - Storme, 2008

$Q^+(2m-1,q)$, $q=p^h$, p prime, has no ovoids if $m>q^2+1$.

Translation ovoids

Let $\mathcal{P}$ be a finite classical polar space.

An ovoid $\mathcal{O}$ of $\mathcal{P}$ is a *translation ovoid* with respect to a point $X \in \mathcal{O}$ if there is a collineation group of $\mathcal{P}$ (called *translation group about* X of $\mathcal{O}$) fixing all totally isotropic lines through X and acting regularly on points of the ovoid different from X.

If (V_1, B_1) and (V_2, B_2) are geometries of the same type then an isomorphism $\alpha : V_1 \to V_2$ is an *isometry* if

$$B_2(u^\alpha, v^\alpha) = B_1(u, v)$$

for all $u, v \in V_1$.

Symplectic groups

Let B be a non-degenerate alternating form on V.

An isometry of V is called *symplectic transformation*, and the group of smplectic transformations is denoted by $\mathrm{Sp}(V)$.

We write $\mathrm{Sp}(n, F)$, $\mathrm{Sp}(n, q)$, etc., for the corresponding groups of matrices.

The kernel of the action of $\mathrm{Sp}(V)$ on $\mathbb{P}(V)$ is

$$Z(\mathrm{Sp}(V)) = \{\pm 1_V\}$$

and we define the *projective symplectic group*

$$\mathrm{PSp}(V) := \mathrm{Sp}(V)/Z(\mathrm{Sp}(V)).$$

When we consider symplectic transformation in $\Gamma\mathrm{L}(V)$ then we get the *(full) projective symplectic group*

$$\mathrm{P\Gamma Sp}(V) := \Gamma\mathrm{Sp}(V)/Z(\Gamma\mathrm{Sp}(V)).$$

Unitary groups

Let B a non-degenerate hermitian form on V.

An isometry of V is called a *unitary transformation*. The set of all unitary transformations of V form a subgroup $\mathrm{GL}(n, q)$ which is called the *unitary group* on V and it is denoted by $\mathrm{U}(n, q)$.

The *full unitary group* $\Gamma\mathrm{U}(V)$ consists of all $\alpha-$semilinear transformations τ of V that induces a collineation of $\mathbb{P}(V)$ that commutes with $\perp$. That is,

$$B(u^\tau, v^\tau) = aB(u, v)^\alpha$$

for some $a \in F$ such that $a = a^q$ and all $u, v \in V$.

The *general unitary group* is $\mathrm{GU}(V) = \Gamma\mathrm{U}(V) \cap \mathrm{GL}(V)$

The kernel of the action of $\mathrm{GU}(V)$ on $\mathbb{P}(V)$ is

$$Z(\mathrm{GU}(V)) = \{c \cdot 1_V : c \in F \textit{ and } c\bar{c} = 1\}$$

and we define the *projective general unitary group*

$$\mathrm{PGU}(V) := \mathrm{GU}(V)/Z(\mathrm{GU}(V))$$

and the *(full) projective unitary group*

$$\mathrm{P\Gamma U}(V) := \Gamma\mathrm{U}(V)/Z(\Gamma\mathrm{U}(V)).$$

Orthogonal groups

Let B a non-degenerate symmetric form on V with polar form Q.

An invertible linear transformation τ of V is said to be *orthogonal* if $Q(\tau v) = Q(v)$, for all $v \in V$.

The set of all orthogonal transformations of V form a subgroup $\mathrm{GL}(n, q)$ which is called the *orthogonal group* on V and it is denoted by $\mathrm{O}(n, q)$.

The *full orthogonal group* $\Gamma\mathrm{O}(V)$ consists of all $\alpha-$semilinear transformations τ of V such that for some $\mathbf{a} \in K$

$$Q(v^\tau) = aQ(v)^\alpha$$

for all $v \in V$.

The *general ortogonal group* is $\mathrm{GO}(V) = \Gamma\mathrm{O}(V) \cap \mathrm{GL}(V)$

The kernel of this action is

$$Z(\mathrm{GO}(n, q)) = \{\pm 1\}$$

and we define the *projective orthogonal group*

$$\mathrm{PGO}(n, q) := \mathrm{GO}(n, q)/Z(\mathrm{GO}(n, q)).$$

and the *projective semilinear orthogonal group*

$$\mathrm{P\Gamma O}(V) := \Gamma\mathrm{O}(V)/Z(\Gamma\mathrm{O}(V)).$$

We have different projective orthogonal groups associated with the three different orthogonal spaces:

$$\mathrm{PGO}^-(2m, q), \quad \mathrm{PGO}(2m+1, q), \quad \mathrm{PGO}^+(2m+2, q),$$

$$\mathrm{P\Gamma O}^-(2m, q), \quad \mathrm{P\Gamma O}(2m+1, q), \quad \mathrm{P\Gamma O}^+(2m+2, q),$$

Translation ovoids

Let $\mathcal{P}$ be a finite classical polar space.

An ovoid $\mathcal{O}$ of $\mathcal{P}$ is a *translation ovoid* with respect to a point $X \in \mathcal{O}$ if there is a collineation group of $\mathcal{P}$ (called *translation group about* X of $\mathcal{O}$) fixing all totally isotropic lines through X and acting regularly on points of the ovoid different from X.

A translation ovoid is called *semilinear* if it has a translation group containing non-linear collineations; it is called *linear* otherwise.

Examples of translation ovoids

Symplectic polar space $\mathcal{W}(3, q)$ *(q even)*

- the elliptic quadric $Q^{-}(3, q)$
- the Suzuki-Tits ovoid ($q = 2^{2h+1}$);

(here the translation group fixes all the tangent lines at $X \in \mathcal{O}$).

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Symplectic polar space $\mathcal{W}(3, q)$ *(*q *even)*

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(here the translation group fixes all the tangent lines at $X \in \mathcal{O}$).

Theorem (Glynn, 1984)

In $\mathcal{W}(3, q)$, q even, linear translation ovoids are either elliptic quadrics or Suzuki-Tits ovoids.

Orthogonal polar spaces

- non-degenerate conics inside $Q^+(3, q)$;
- ovoids of $Q^+(5, q)$ corresponding to semifield spreads;
- ovoids of $Q(4, q)$ corresponding to symplectic semifield spreads.

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- ovoids of $Q(4, q)$ corresponding to symplectic semifield spreads.

Theorem (Cardinali - Lunardon - Polverino - Trombetti, 2002)

In $Q(4, 2^h)$ linear translation ovoids are elliptic quadrics.

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- ovoids of $Q(4, q)$ corresponding to symplectic semifield spreads.

Theorem (Cardinali - Lunardon - Polverino - Trombetti, 2002)

In $Q(4, 2^h)$ linear translation ovoids are elliptic quadrics.

Theorem (Lunardon - Polverino, 2004)

$Q^+(3, q)$, $Q(4, q)$ and $Q^+(5, q)$ are the only finite orthogonal spaces containing a linear translation ovoids.

What about translation ovoids of unitary polar spaces?

Translation ovoids in $\mathcal{H}(2m-1,q^2)$

A non-isotropic line of $\mathrm{PG}(2m-1,q^2)$ which intersects $\mathcal{H}(2m-1,q^2)$ in more then one point is called a *hyperbolic line*.

An ovoid $\mathcal{O}$ is called *locally hermitian* with respect to a point $X \in \mathcal{O}$ if $\mathcal{O}$ is the union of q^{2n} hyperbolic lines through X.

Bader - Trombetti, 2004

Every linear translation ovoid of $H(3,q^2)$ is locally hermitian.

The connections with translation ovoids of $\mathcal{H}(3, q^2)$ and semifield spreads are intertwined (via the Shult Embedding) with Shult sets (E. Shult, 2005).

Indicator sets

$\mathrm{AG}(2, q^2)$ a finite Desarguesian affine plane
ℓ_∞ the line at infinity of $\mathrm{AG}(2, q^2)$.
$\mathrm{PG}(2, q^2) = \mathrm{AG}(2, q^2) \cup \ell_\infty$

A subset $\mathcal{F}$ of the point-set of $\mathrm{AG}(2, q^2)$ is called a *indicator set* if:

(i) $|\mathcal{F}| = q^2$

(ii) there exists a Baer subline H of ℓ_∞ such that any secant line $\mathcal{F}$ meets ℓ_∞ in a point not in H.

Shult sets

$\mathrm{PG}(2, q^2) = \mathrm{AG}(2, q^2) \cup \ell_\infty$

H^* a Baer pencil of lines whose center P is an affine point.

A subset $\mathcal{S}$ of the line-set of $\mathrm{PG}(2, q^2)$ is called a *Shult set* if:

(i) $|\mathcal{S}| = q^2$

(ii) no line of $\mathcal{S}$ pass through P

(iii) every pair of distinct lines of $\mathcal{S}$ intersect at a point not in H^*.

Under duality * in $\mathrm{PG}(2, q^2)$, any indicator set $\mathcal{F}$ w.r.t. H gives a Shult set in $\pi = \mathrm{PG}(2, q^2)^*$.

Shult embedding

Let $\mathcal{S}$ be a Shult set in π w.r.t H^*

- $\mathrm{PG}(3, q^2)$ containing a Hermitian polar space $\mathcal{H}(3, q^2)$
- $\pi \hookrightarrow \mathrm{PG}(3, q^2)$ in such a way that $\mathcal{H}(3, q^2) \cap \pi = H^*$.

Then

$$O(\mathcal{S}) = \{L^\perp \cap \mathcal{H}(3, q^2) : L \in \mathcal{S}\}$$

is a locally hermitian ovoid

Examples of indicator sets

Classical examples

1. $\mathcal{F}$ is any affine line of $\mathrm{AG}(2, q^2)$ with point at infinity not in H (classical case)
2. $\mathcal{F}$ is any affine Baer subplane of $\mathrm{AG}(2, q^2)$ whose set of points at infinity is disjoint from H (semi-classical case)

Examples by Cossidente - Ebert - Marino - AS., 2006

3. $\mathcal{F}_T = \{(1, a, \mathrm{Tr}(a)) : a \in \mathrm{GF}(q^2)\}$, q odd (Trace type)
4. $\mathcal{F}_B = \{(1, a, a^{p^f}) : a \in \mathrm{GF}(q^2)\}$, where $q = p^{2e}$ is odd, $f|e$ (Frobenius type)

The above examples are all the known (linear) translation ovoids up to now.

Theorem (Johnson, 2007)

The Trace type ovoid corresponds to a class of Kantor-Knuth semifield flock spread; the Frobenius type ovoid corresponds to a subclass of the semifields of Hughes- Kleinfeld.

What about translation ovoids of $\mathcal{H}(2m-1, q^2)$, $m \geq 3$?

Let $(\mathbf{e}_1, \ldots, \mathbf{e}_m, \mathbf{f}_1, \ldots, \mathbf{f}_m)$ be a basis consisting of mutually orthogonal hyperbolic pairs $(\mathbf{e}_i, \mathbf{f}_i)$, $i = 1, \ldots, m$, so that

$$\mathcal{H}(2m-1, q^2) : X^q Y - XY^q + \overline{\mathbf{X}}\mathbf{Y}' - \mathbf{X}'\overline{\mathbf{Y}} = 0;$$

here:

$\mathbf{a} := (a_1, \ldots, a_{m-1})$
$\mathbf{a}'$ is the transpose of $\mathbf{a}$
$\overline{\mathbf{a}} := (a_1^q, \ldots, a_{m-1}^q)$

Recall that the automorphism group of $\mathcal{H}(2m-1, q^2)$ is

$$\mathrm{P\Gamma U}(2m, q^2) = \mathrm{PGU}(2m, q^2) \rtimes \mathrm{Aut}(\mathrm{GF}(q^2)).$$

Let $\mathcal{O}$ be a translation ovoid of $\mathcal{H}(2m-1, q^2)$ with translation group G around P.
As $\mathrm{PGU}(2m-1, q^2)$ is transitive on points of $\mathcal{H}(2m-1, q^2)$ we can assume $P = \langle \mathbf{e}_1 \rangle$.

Lemma

Let E be the subgroup of $\mathrm{PGU}(2m, q^2)$ fixing P, leaving invariant all totally isotropic lines through P and acting regularly on isotropic points not in $P^\perp$. Then the generic element of E has the following $2m \times 2m-$matrix form

$$\begin{pmatrix} 1 & -\overline{\mathbf{a}} & c - \overline{\mathbf{a}}\mathbf{b}' & \overline{\mathbf{b}} \\ \mathbf{0}' & I_n & \mathbf{b}' & 0_n \\ 0 & \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0}' & 0_n & \mathbf{a}' & I_n \end{pmatrix}$$

where $c \in \mathrm{GF}(q)$.

We represent E as

$$\{[\mathbf{a}, \mathbf{b}, c] : \mathbf{a}, \mathbf{b} \in \mathrm{GF}(q^2)^{m-1}, c \in \mathrm{GF}(q)\}$$

with

$$[\mathbf{a}_1, \mathbf{b}_1, c_1] * [\mathbf{a}_2, \mathbf{b}_2, c_2] = [\mathbf{a}_1 + \mathbf{a}_2, \mathbf{b}_1 + \mathbf{b}_2, c_1 + c_2 + \overline{\mathbf{a}}_2 \mathbf{b}_1' + \mathbf{a}_2 \overline{\mathbf{b}}_1'].$$

Then:

- $K = \{[\mathbf{0}, \mathbf{0}, c] : c \in \mathrm{GF}(q)\}$ is an elementary abelian subgroup of order q and it fixes every hyperbolic line of $\mathcal{H}(2m-1, q^2)$ at P;
- E/K is an elementary abelian group of order $q^{4(m-1)}$.
- if $g \in E$ fixes one hyperbolic line through P then $g \in K$.

Let $\mathrm{Aut}(GF(q^2)) = \langle\varphi\rangle$ where

$$\begin{array}{rccc} \varphi: & \mathrm{GF}(q^2) & \longrightarrow & \mathrm{GF}(q^2) \\ & x & \longmapsto & x^p \end{array}$$

Then every φ^j, $j = 1, \ldots, 2h$, induces the collineation

$$\begin{array}{rccc} \Phi^j: & \mathrm{PG}(2m-1, q^2) & \longrightarrow & \mathrm{PG}(2m-1, q^2) \\ & (X, \mathbf{X}, Y, \mathbf{Y}) & \longmapsto & (X^{\varphi^j}, \mathbf{X}^{\varphi^j}, Y^{\varphi^j}, \mathbf{Y}^{\varphi^j}); \end{array}$$

here $\mathbf{a}^{\varphi^j} = (a_1^{p^j}, \ldots, a_{m-1}^{p^j})$.

The action of Φ^j *in* $\mathrm{PG}(n, q^2)$

- Φ^j fixes P and preserves $\mathcal{H}(2m-1, q^2)$ setwise
- $\mathrm{Fix}(\Phi^j)$ is the canonical subgeometry $\mathrm{PG}(2m-1, p^m)$ generated by $(\mathbf{e}_1, \mathbf{f}_1, \ldots, \mathbf{e}_m, \mathbf{f}_m)$ over $\mathrm{GF}(p^s)$ for $s = GCD(j, 2h)$.

Theorem (King - AS, 2012)

If $m > 2$ then $H(2m-1, q^2)$ has no semilinear translation ovoids.

Representation of unitary spaces over $\mathrm{GF}(q)$

Lemma (AS, 2007)

If $\mathcal{O}$ is a (linear) translation ovoid with respect to P of $\mathcal{H}(2m-1, q^2)$, with translation group G, then $K \leq G$ and $\mathcal{O}$ is locally hermitian.

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If $m \geq 3$, then every translation ovoid of $\mathcal{H}(2m-1, q^2)$ is locally hermitian.

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...and we can use the Barlotti-Cofman representation of $\mathrm{PG}(2m-1, q^2)$ into $\mathrm{PG}(4m-1, q)$ with respect to a fixed hyperplane.

To see what $\mathcal{H}(2m-1,q^2)$ is in $\mathrm{PG}(4m-1,q)$ we take the fixed hyperplane to be $P^\perp$.

Proposition (Lunardon, 2006; AS, 2012)

$\mathcal{H}(2m-1,q^2)$ is represented as a cone of $\mathrm{PG}(4m-1,q)$ projecting a hyperbolic quadric $\mathcal{Q}^+(4m-3,q)$ from a point.

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Theorem (King - AS, 2012)

Every (linear) translation ovoid of $\mathcal{H}(2m-1,q^2)$ determines a linear translation ovoid of $\mathcal{Q}^+(4m-3,q)$.

Combining the previous theorem and the result of Lunardon and Polverino on finite orthogonal polar spaces with linear translation ovoids we get the following

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Theorem (King - AS, 2012)

The only finite unitary polar space having translation ovoids is $\mathcal{H}(3, q^2)$.

Semilinear translation ovoids of $\mathcal{H}(3, q^2)$

Let $\mathcal{O}$ be a translation ovoid of $\mathcal{H}(3, q^2)$ with translation group G around P.

Then $|\mathcal{O}| = q^3 + 1$ and $G \leq \mathrm{P\Gamma U}(4, q^2)$ has order q^3.

Examples

- non-degenerate hermitian curves
- infinite families of translation ovoids of $\mathcal{H}(3, q^2)$ (CEMS, 2006)

All the exhibited translation ovoids are linear and thus locally hermitian.

$$E = \{[a,b,c] : a,b \in \mathrm{GF}(q^2), c \in \mathrm{GF}(q)\}$$

$$[a_1,b_1,c_1] * [a_2,b_2,c_2] = [a_1+a_2, b_1+b_2, c_1+c_2+\overline{a}_2 b_1 + a_2 \overline{b}_1].$$

Set $\phi = \Phi^h$. Then

- $W := \langle E, \phi \rangle \leq \mathrm{P\Gamma U}(4,q^2)$
- $|W| = 2q^5$
- W fixes P, fixes every totally isotropic line through P and acts transitively on points of $\mathcal{H}(3,q^2) \setminus P^{\perp}$.

Any subgroup of W acting regularly on points of $\mathcal{H}(3, q^2) \setminus P^{\perp}$ (*elation group around P*) has order q^5.

When q is odd, E is the unique Sylow $p-$subgroup of W.

Theorem

If q is odd, then every translation ovoid of $\mathcal{H}(3, q^2)$ is linear and so locally hermitian.

Translation ovoid of $\mathcal{H}(3, q^2)$, q *even*

Some comment

- there are many (inequivalent) elation groups around P (classified by R.L. Rostermundt, 2007)
- there are many subgroups of W of order q^5.
- W contains elements that are not elations (ϕ for example)
- an elation group about P is a subgroup of W
- the translation group G of an ovoid is also a subgroup of W.
- it is not immediately clear that G is a subgroup of an elation group about P.

$\mathcal{H}(3, q^2)$ contains a symplectic polar spaces $\mathcal{W}(3, q)$.

An *i-tight set* $\mathcal{T}$ of $\mathcal{H}(3, q^2)$ is a set of points such that every point in $\mathcal{T}$ is collinear with $q^2 + i$ points of $\mathcal{T}$, while every point not in $\mathcal{T}$ is collinear with i points of $\mathcal{T}$.

Lemma

Every $\mathcal{W}(3, q)$ contained in $\mathcal{H}(3, q^2)$ is a $(q+1)-$tight set.

Proposition

Let $\mathcal{O}$ be any ovoid of $\mathcal{H}(3, q^2)$ and $\mathcal{T}$ a symplectic subgeometry contained in $\mathcal{H}(3, q^2)$. Then $\mathcal{O}$ and $\mathcal{T}$ intersect in $q + 1$ points.

Let $q = 2^h$. We have

- the derived group of W is $W' = \{[a, b, c] : a, b, c \in \mathrm{GF}(q)\}$
- W/W' is a vector space of dimension $2h + 1$ over $\mathrm{GF}(2)$
- (Since every $p-$group is nilpotent and W is a 2-group, it follows that) W' is a subgroup of the Frattini subgroup $\mathcal{F}(W)$
- Each maximal subgroup of W has order q^5 and contains W'
- $K \leq W'$.

Lemma

If $G \leq \mathrm{P\Gamma U}(4, q^2)$ is a translation group of an ovoid, then GW' is a maximal subgroup of W and $|G \cap W'| = q$.

Lemma

If $\mathcal{O}$ is semilinear then the previous lemma does not hold.

Theorem (King - AS, 2012)

If q is even, then every translation ovoid of $\mathcal{H}(3, q^2)$ is linear.

Corollary (King - AS, 2012)

Every translation ovoid of $\mathcal{H}(3, q^2)$ is locally hermitian.