k-sets in PG(3,q) of type (m, n) with respect to planes

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- $\mathbf{P} = PG(d, q)$
- $1 \le h \le d-1$ $\mathcal{P}_h =$ *the family of all h-dimensional subspaces of* $\mathbf{P}$
- $0 \le m_1 < \ldots < m_s$

$K \subseteq P$ is of ***class*** $[m_1, \ldots, m_s]_h$ *if* $|K \cap \Pi| \in \{m_1, \ldots, m_s\}$ *for all* $\Pi \in \mathcal{P}_h$

$m_1, \ldots, m_s$ are the ***intersection numbers*** of K

K is of ***type*** $(m_1, \ldots, m_s)_h$ *if for every intersection number* m_j *there is a subspace* $\Pi \in \mathcal{P}_h$ *such that*

$$|K \cap \Pi| = m_j$$

k-sets in P via intersection numbers

- ***CHARACTERIZATION PROBLEM*** (B. Segre point of view in (finite) geometry)

- ***EXISTENCE PROBLEM*** (codes theory, strongly regular graphs)

- ***CLASSIFICATION PROBLEM*** (for small values of q)

A (q+1)-set of PG(2, q), q odd, of type $(0, 1, 2)_1$ is a (non-degenerate) conic. (B. Segre 1954)

A (q^2+1)-set of PG(3, q), q odd, of type $(0, 1, 2)_1$ is an elliptic quadric. (A. Barlotti 1955, Panella 1955)

*A k-set of **P** of type $(m, n)_{d-1}$ spanning **P** gives rise to a two weight [k, d+1] code with weigths k – m, k-n,*

and

*a k-set of **P** of type $(m, n)_{d-1}$ spanning **P** derives from a two weight [k, d+1] code with weigths k – m, k-n*

Intersection with lines

K *is of class* $[0, 1, q+1]_1 \Leftrightarrow$ K *is a subspace of* **P**

K *is of type* $(m)_1 \Leftrightarrow$ *K is the empty set* $(m = 0)$ *or* **P** $(m = q+1)$

K is of type $(0,1)_1 \Leftrightarrow$ *K is a point of* **P**

There is no k-set of type $(0, q+1)_1$

Intersection with lines

d $\geq$ 3, K of type $(m, n)_1$ in **P**, S a *h*-dimensional subspace of **P** :

S $\cap$ K *is of type* $(m, n)_1$ *in* **PG**(h, q)

k-sets in PG*(2, q)*

$$K \text{ of type } (0,n),\ n \le q \;\Rightarrow\; n \mid q \text{ and } k = qn - q + n \quad \text{(maximal arc)}$$

A hyperoval is a maximal arc with n = 2.

If n > 2, for every pair (n, q) = (2^a, 2^b), 0 < a < b , there are maximal arcs (Denniston (1969); Thas (1974), (1980); Mathon (2002))

*“The most wanted research problem”**

Conjecture (J. A. Thas 1975): ***For q odd there is no maximal arc***

The conjecture is true (Ball, Blokhuis, Mazzocca 1997 in Combinatorica)

* T. Penttila, G.F. Royle “Sets of type (m,n) in the affine and projective planes of order 9” (1995 in Designs Codes and Cryptography).

k-sets in **PG**$(2, q)$

K a k-set of type $(1, n) \Rightarrow q = p^{2h}$, $q = (n\text{-}1)^2$ *and K is a Baer subplane or a Hermitian arc.*

$2 \leq m < n < q+1$:

K a k-set of type $(m, m+s)$, $s^2 \geq q$, $(m, m+s) = (m - 1, m+s - 1) = 1 \Rightarrow q = s^2$ *and* $k = m(s^2 + s + 1)$ *or* $k = s^3 + s(s\text{-}1)(m\text{-}1)+m$

A k-set in **PG**$(2, q^2)$ *of type* $(m, m+q)$ *with* $k = m(q^2+q+1)$

The set of points of the union of m pairwise disjoint Baer subplanes

$m < q^2-q+1$

k-sets in PG(*2, q*)

K a k-set of type (m, n) in PG(2, q)

$(2 \leq m < n \leq q-1)$

$$\Downarrow$$

- $k^2 - k[1+(q+1)(n+m-1)]+mn(q+1)(q^2+1)=0$
- $n - m \mid q$
- $m\,q + n \leq k \leq (n-1)q + m$

Intersection with lines (d $\geq$ 3)

A k-set of **P**, $d \geq 3$, *of type* $(0, n)_1$, $n \leq q$, *either is a point* ($n=1$) *or* **P** *less a hyperplane* ($n = q$). (M. Tallini Scafati 1969)

A k-set of **P**, $d \geq 3$, *of type* $(m, q+1)_1$ *is* **P** *less a point* ($m = q$) *or a hyperplane* ($m = 1$).

Intersection with lines

$d \geq 3 \quad n \geq 2, \; m > 0$:

K is a k-set of type $(1, n)_1$ of P ⇒ K is a hyperplane

Proof :

- **H = K∩S_3 is a k-set of type $(1,n)_1$ in PG(3, q)**

Intersection with lines (d ≥ 3)

- **PG(3,q) has no set of type $(1, n)_1$ and $n < q$** (***PG(2,q) plays a special rôle (as in Tallini Scafati (1969))***)

- **P has no set of type $(1, q)_1$**

K is a k-set of type $(m, q)_1$ of **P** ⇒ *K is* **P** *less a hyperplane*

Intersection with lines (d = 3)

K a k-set of type $(m, n)_1$ *in* $PG(3, q)$

$$(2 \leq m < n \leq q-1)$$

$$\Downarrow$$

- *q is a odd square*
- $k = [1 + (q^2+1)(q + \varepsilon \cdot q^{1/2}) \pm q \cdot q^{1/2}]/2$
  $(\varepsilon = \pm 1)$
- $m = [q+1- q^{1/2}(1- \varepsilon)]/2$
- $n = [q+1+ q^{1/2}(1- \varepsilon)]/2$

Intersection with lines ($d \geq 3$)

S a 3-dimensional subspace of P K of type $(m, n)_1$ in P

$S \cap K$ is of type $(m, n)_1$ in PG(3, q) $\Rightarrow$

- $m = [q+1- q^{1/2} (1- \varepsilon)] /2$
- $n = [q+1+ q^{1/2} (1- \varepsilon)] /2$
- $k = [1 +(q^{d-1}+..+ q+1)(q + \varepsilon \cdot q^{1/2}) \pm (q^{1/2})^{d}] /2$

  $(\varepsilon = \pm 1)$

Intersection with lines (d ≥ 3)

Characterizations of Quadrics and Hermitian vareties as sets of class $[0, 1, n, q+1]_1$ *with some extra regularity condition (e.g. at each point p the set of* 1-*secant lines is a subspace, on each point there is at least one n-line):*

quadratic sets and n-varieties.

Intersection with lines

K with set of line-intersection numbers

$\mathscr{I} = \{0, 1, \ldots, s\}$

$m = \min \mathscr{I} \setminus \{0\}$ and $n = \max \mathscr{I} \setminus \{0\}$

b_i = # i-secant lines, $i \in \{0, 1, \ldots, s\}$

$\Theta_r = q^r + q^{r-1} + \ldots + q + 1$

Then

Intersection with lines (d ≥ 2)

$b_0 \geq [k2 - k(1+(m+n-1)\,\Theta_{r-1}) + mn\,\Theta_r\,\Theta_{r-1} / \Theta_1]/mn$

with equality iff K is of class $[0, m, n]_1$.

moreover

b_0 *and the above ratio are both* = 0 *iff* K *is of type* $(m, n)_1$

k-sets of type $(m, n)_h$ in $\mathcal{P}$

A k-set of type $(m, n)_{\mathrm{h}}$, $h \leq d\text{-}2$ *is of type* $(r, s)_{\mathrm{i}}$ *for every* $i \in \{h+1, \ldots, d\text{-}1\}$

K a k-set of type $(m, n)_{\mathrm{d-1}} \Rightarrow n - m \mid q^{d-1}$ [Tallini Scafati 1969]

K a q^t*-set* $(m, n)_{\mathrm{d-1}}$ *is either a point or* $\mathcal{P}$ *less a hyperplane* [L. Berardi – T. Masini On sets of type $(\mathrm{m,n})_{\mathrm{r-1}}$ in PG(r,q) Discrete Math 2009]

k-sets of type $(m, n)_2$ in $PG(3, q)$

- *Hyperbolic quadrics,*
- *(q^2+1)-caps*
- *non- singular Hermitian varieties,*
- *subgeometries*
- *sets of points on m pairwise skew lines*
- *any subset of the ovoidal partition of $PG(3, q)$*
- *a subgeometry G union a family of pairwise skew lines external to G*

$(n - m = q)$

k-sets of type $(m, n)_2$ in PG(3, q)

J.A. Thas (1973):

The only sets in ***P*** *of type (1, n) w.r. to hyperplanes are the lines or the ovoids of PG(3, q) .*

The proof uses an algebraic argument.

Such result has been proved in a more general setting and in a geometric way:

k-sets of type $(m, n)_2$ in PG(3, q)

N. Durante, V.N., D. Olanda (2002):

(S = a 3-dimensional locally projective planar space of order q)

$K \subseteq S$ and meeting every plane in either 1 or n ($n > 1$) points is a line (with $q + 1$ points) or a set of $q^2 + 1$ points no three of which are collinear.

k-sets of type $(m, n)_2$ in $PG(3, q)$

N. Durante, D. Olanda (2006):

(S = a 3-dimensional locally projective planar space of order q)

A set K of points of S meeting every plane in either 2 or n (n >2) points is a pair of skew lines (both of size q + 1)

k-sets of type $(m, n)_2$ in $PG(3, q)$

O. Ferri (1980):

K cap of type $(m, n)_2$ in $PG(3, q)$

$\Downarrow$

K is an ovoid ($m=1$) or $q=2$ $m = 0$ and K is $PG(3, 2)$ less a plane

k-sets of class $[3, n]_2$ in $PG(3, q)$

The union of three pairwise skew lines in $PG(3, q)$,

A plane in $PG(3, 2)$

$PG(3, 2)$

$PG(3, 2)$ embedded in $PG(3, 4)$

k-sets of type $(3, n)_2$ in $PG(3, q)$

- $q > 2 \;\Rightarrow (n - 3) \mid q$ (i.e. $n \leq q + 3$)
- either $n = q + 3$ or $s \leq 3$ for each s-line

k-sets of type $(3, q+3)_2$ in PG(3, q)

V.N. - D. Olanda (2012): n = q+3

- ***If K contains no line then q =3 or 4.***
- ***If q = 4 then K = PG(3, 2).***
- ***If q =3 then k = 12 o k = 15 and K is one of the following three Examples:***

k-sets of type $(3, 6)_2$ in PG(3, 3)

K_1 ($k = 12$)
A (1 0 0 0) , B(0 1 0 0), C (0 1 1 1), D (0 0 1 0),
E (0 1 0 1), F (0 0 0 1), G (1 0 0 1), H (1 1 0 1) ,
I (1 0 2 0), L (1 2 2 0), M (1 0 2 1), N (0 1 1 0)

K_2 ($k = 15$)
A (1 1 2 1) , B(1 0 0 0), C (0 1 0 0), D (0 0 1 0),
E (0 0 0 1), F (0 0 1 2), G (1 1 1 1), H (1 1 1 2) ,
I (1 0 2 0), L (1 2 2 0), M (0 1 2 2), N (0 1 1 0),
O(1 0 2 2) ,P(1 2 1 1), Q(1 2 1 2)

k-sets of type $(3, 6)_2$ in PG(3, 3)

K_3 ($k = 15$)

A (1 0 0 0) , B(0 1 1 0), C (0 1 0 0),
D (0 0 1 0), E (0 0 0 1), F (1 1 2 1),
G (1 1 1 1), H (1 0 1 2) , I (1 1 1 2),
L (1 2 2 0), M (0 1 2 2), N (1 1 2 2),
O(0 1 2 1) ,P(1 0 1 1), Q(1, 0, 2, 0)

k-sets of type $(3, 6)_2$ in PG(3, 3)

K_1: *[12, 4, 9]$_3$-code with second weight 9*

("subcode" of the ternary extended Golay code)

K_2 *and* K_3: *two different [15, 4, 6]$_3$-codes with second weight 12*

K_1 K_2 and K_3 : an exhaustive research obtained by adapting a program in MAGMA contained in

[S. Marcugini, F. Pambianco, *Minimal 1-saturating sets in PG(2, q), q ≤ 16,* Austral. J. Combin. 28 (2003), 161-169]

k-sets of type $(3, q+3)_2$ in PG(3, q)

V.N. - D. Olanda (2012): $n = q+3$

If K contains a line then K is the set of the points of the union of three skew lines.

k-sets of type $(3, n)_2$ in $PG(3, q)$

$n < q+3$:

- *there is a 3-line L s.t. all planes on L are h- plane :*

  $q = 8$, $n = 7 = q/2 + 3$ and $k=39$.

k-sets of type $(3, n)_2$ in PG(3, q)

- *on each 3-line there is at least one 3-plane :*

either

there is a n-plane with a point on no 2-line and $q = 2^t$, $n = 2^s + 3$, $2 \leq s \leq t-1$

or

$$(q+1)n-4q \leq k \leq qn - 3q + 3$$

k-sets of type $(m, n)_2$ in PG(3, q)

$$m \leq 3 \Rightarrow n \leq m + q \text{ and } k \geq m(q+1)$$

$$m \leq 3 \Rightarrow m \leq q+1$$

k-sets of type $(m, n)_2$ in PG(3, q) with $m \leq q+1$

K_1 be the only 12-set of type $(3, 6)_2$ in PG(3, 3)

L external line to K_1 : $\Omega = K_1 \cup L$
is a a 16-set of type $(4, 7)_2$ in PG(3, 3):
$[16, 4, 9]_3$ code with second weight 12

k-sets of type $(m, n)_2$ in PG(3, q) with $m \leq q+1$

Planes: $m = q+1$ $k = q^2+q+1 < m(q+1)$

$m \leq q \Rightarrow k \geq m(q+1)$

Theorem (V.N. 20??)

A k-set K in PG(3,q) of type $(q+1, n)_2$, $m \leq q+1$ is a plane or $k \geq (q+1)^2$ and at least one external line exists. If $k = (q+1)^2$ and K contains at least $q - q^{1/2}$ pairwise skew lines then either

k-sets of type $(m, n)_2$ in PG(3, q) with $m \leq q+1$

K is the set of points of q+1 pairwise skew lines or $q = s^2$ and K is the set of points of PG(3,s) union the points of $s^2 - s$ pairwise skew lines.

k-sets of type $(m, n)_2$ in PG(3, q) with $m \leq q+1$

$\Omega = K_1 \cup L$ the 16-set of type $(4, 7)_2$ in PG(3, 3):

$4 = 3 + 1 = q + 1$ and $16 = m(q+1) \Rightarrow$ an external line M to exists Ω

$\Omega \cup M$ is a 20-set of type $(5, 8)_2$ in PG(3, 3)

$[20, 4, 12]_3$ code with second weight 15

k-sets of type $(m, n)_2$ in PG(3, q) with $m \leq q$

Theorem

Let K be a set of points of PG(3, q) of type $(m, n)_2$ with $m \leq q$ and $k = m(q+1)$. If $s \geq m$ for every s-line with $s \geq 3$ then either K is the set of points of m pairwise skew lines, or $q=(m-1)^2$ and K is the subgeometry PG(3, m-1) or $m = q = 3$ and K is one of the sets K_1 and K_2.

k-sets of type $(m, n)_2$ in PG(3, q) with $m \le q$

Theorem

A k-set K in PG(3,q) of type $(m, n)_2$ $m \le q$ is and $k = m(q+1)$. If contains at least $q - q^{1/2} - 1$ pairwise skew lines then either K is the set of points of m pairwise skew lines or $m = q = s^2$ and K is the set of points of PG(3,s) union the points of $s^2 - s - 1$ pairwise skew lines.

Hermitian variety in PG(3, q^2)

A non-singular Hermitian variety H(3, q^2) in PG(3, q^2) is of type $(1, q+1, q^2+1)_1$ and $(q^3+1, q^3+q^2+1)_2$

L. Berardi – T. Masini On sets of type $(m,n)_{r-1}$ in PG(r,q) Discrete Math 2009:

A k-set of type $(m,n)_2$ in PG(3, q^2) is of *Hermitian type* if $k = q^3+q^2+q+1$, $m = q^3+1$, $n = q^3+q^2+1$

Hermitian variety in $PG(3, q^2)$

Theorem (Berardi-Masini 2009)

A (q^3+q^2+q+1)-set of type $(m, n)_2$ in $PG(3, q^2)$ is of Hermitian type.

J. Schillewaert-J.A.Thas *Characterizations of hermitian varieties by intersection numbers* Designs Codes and Cryptography (2008):

Hermitian variety in $PG(3, q^2)$

Theorem (BSchillewaert-J.Thas 2008)
A k-set of types $(1, q+1, q^2+1)_1$ and $(q^3+1, q^3+q^2+1)_2$ in $PG(3, q^2)$ is a Hermitian variety $H(3, q^2)$

Moreover, they characterize $H(d, q^2)$ with respect planes and solids for any dimension $d \geq 4$

(First solve the case $d = 4$, then study $K \cap S$ and $K \cap T$ with S, T a 3-space and a 4-space of $\mathcal{P}$ respectively)

Hermitian variety in $PG(3, q^2)$

Theorem (V.N.20??) *Let K be a $m(q+1)$-set of $PG(3, q)$, of types $(1, s+1, q+1)_1$ and $(m, n)_2$, $1 \leq s \leq q-1$, then $q = s^2$ and K is a Hermitian variety $H(3, q^2)$*