Semi-resolving sets for $\mathrm{PG}(2, q)$

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Feedback

Let $\Pi_q = (\mathcal{P}, \mathcal{L})$ be a projective plane of order q.

Idea (Bailey, BCC 2011): if $\mathcal{P}_S$ is a point-set that resolves all lines, and $\mathcal{L}_S$ is a line-set that resolves all points, then $\mathcal{S} = \mathcal{P}_S \cup \mathcal{L}_S$ is clearly a resolving set.

Such a resolving set is called a **split resolving set**; its parts, $\mathcal{P}_S$ and $\mathcal{L}_S$ are called **semi-resolving sets**. Note that if the plane is self-dual (like $\mathrm{PG}(2, q)$), then we may assume that a semi-resolving set resolves the lines of the plane.

Definition

The size of the smallest split resolving set for Π_q is $\mu^*(\Pi_q)$.
The size of the smallest semi-resolving set for Π_q is $\mu_S(\Pi_q)$; well-defined if Π_q is self-dual.

Clearly $\mu(\Pi_q) \leq \mu^*(\Pi_q)$, and $\mu^*(\mathrm{PG}(2, q)) = 2\mu_S(\mathrm{PG}(2, q))$.

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Semi-resolving sets for projective planes

P is a point, ℓ is a line;

- $d(P, \ell) = 1 \iff P \in \ell$
- $d(P, \ell) = 3 \iff P \notin \ell$

There is no third possibility: incidence determines distance. I.e., a distance list of ℓ with respect to a point-set $\mathcal{S} \leftrightsquigarrow \ell \cap \mathcal{S}$.

So $\mathcal{S}$ is a semi-resolving set $\iff$ $\mathcal{S} \cap \ell$ is unique for every line ℓ.

Clear: $|\ell \cap \mathcal{S}| \geq 2 \Rightarrow \mathcal{S}$ resolves ℓ.

So a point-set $\mathcal{S}$ is a semi-resolving set if and only if

- there is at most one skew line to $\mathcal{S}$
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Constructions

Definition

A t-fold blocking set is a set of points that intersects every line in at least t points.

Blocking set $=$ 1-fold blocking set

Double blocking set $=$ 2-fold blocking set

A double blocking set is clearly a semi-resolving set.

τ_2: the size of the smallest double blocking set.

Hence we have $\mu_S \leq \tau_2$. (Bailey)

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Constructions

$\mathcal{B}$ a double blocking set, $P \in \mathcal{B}$ arbitrary.

Then $\mathcal{S} = \mathcal{B} \setminus \{P\}$ is a semi-resolving set:

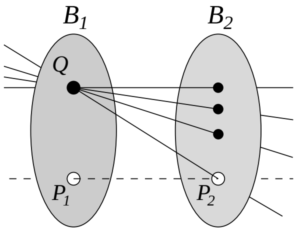
- there is no skew line to $\mathcal{S}$;
- there is at most one tangent line through any point Q of $\mathcal{S}$: PQ may be tangent.

So we have $\mu_S \leq \tau_2 - 1$. (Bailey)

Constructions

$\mathcal{B}_1$, $\mathcal{B}_2$ disjoint blocking sets, $P_1 \in \mathcal{B}_1$, $P_2 \in \mathcal{B}_2$ arbitrary.

Then $\mathcal{S} = \mathcal{B}_1 \setminus \{P_1\} \cup \mathcal{B}_2 \setminus \{P_2\}$ is a semi-resolving set:



- there is at most one skew line to $\mathcal{S}$: P_1P_2 may be skew
- there is at most one tangent line through any point $Q \in \mathcal{S}$: say, $Q \in \mathcal{B}_1$; every line through Q intersects $\mathcal{B}_2$, except possibly QP_2.

Results

If q is a square, we find disjoint Baer subplanes (blocking sets of size $q + \sqrt{q} + 1$). Thus we have $\mu_S(\mathrm{PG}(2,q)) \leq 2q + 2\sqrt{q}$.

Aart Blokhuis (unpublished): $\mu_S(\Pi_q) \geq 2q + \sqrt{2q}$ (roughly).

We prove: if $q \geq 87$, then $\mu_S(\mathrm{PG}(2,q)) \geq 2q + 2\sqrt{q}$. In fact:

Theorem

Let $\mathcal{S}$ be a semi-resolving set for $\mathrm{PG}(2,q)$, $q \geq 4$. If $|\mathcal{S}| < 9q/4 - 3$, then one can add at most two points to $\mathcal{S}$ to obtain a double blocking set; thus $|\mathcal{S}| \geq \tau_2 - 2$.

Corollary

Let $\mathcal{S}$ be a semi-resolving set for $\mathrm{PG}(2,q)$, $q \geq 4$. Then $|\mathcal{S}| \geq \min\{9q/4 - 3, \tau_2(\mathrm{PG}(2,q)) - 2\}$.

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Results

- If $q \geq 9$ is a square, then $\tau_2 = 2q + 2\sqrt{q} + 2$. Thus the corollary gives $\mu_S \geq \tau_2 - 2$ if $q \geq 87$.
- If $q = r^h$, r odd, $h \geq 3$ odd, then $\tau_2 \leq 2(q-1)/(r-1)$. Thus the corollary gives $\mu_S \geq \tau_2 - 2$ if $r \geq 11$.

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Blocking semiovals

Definition

A point-set $\mathcal{S}$ is a ***semioval****, if it has precisely one tangent at each of its points. A* ***blocking semioval*** *is a semioval that is a blocking set.*

Theorem (Dover)

Let $\mathcal{S}$ be a blocking semioval in an arbitrary projective plane of order q. If $q \geq 7$, then $|\mathcal{S}| \geq 2q + 2$. If $q \geq 3$ and there is a line intersecting $\mathcal{S}$ in $q - k$ points, $1 \leq k \leq q - 1$, then $|\mathcal{S}| \geq 3q - 2q/(k + 2) - k$.

A corollary for blocking semiovals

Corollary

Let $\mathcal{S}$ be a blocking semioval in $\mathrm{PG}(2, q)$, $q \geq 4$. Then $|\mathcal{S}| \geq 9q/4 - 3$.

Proof.

A blocking semioval $\mathcal{S}$ is a semi-resolving set. Suppose $|\mathcal{S}| < 9q/4 - 3$. Then $|\mathcal{S}| \geq \tau_2 - 2 > 2q + 1$, so $\mathcal{S}$ has at least $2q + 2$ tangents, but two points may block at most $2q + 1$ of them. □

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$\mathcal{S}$ is a semi-resolving set, P is a point.

Let $|\mathcal{S}| = 2q + \beta$. Homework: $\beta \geq -1$.

- $\mathrm{ind}_i(P)$: the number of i-secants to $\mathcal{S}$ through P
- $\mathrm{ind}(P) := 2\mathrm{ind}_0(P) + \mathrm{ind}_1(P)$ (index)

We show that $\mathrm{ind}(P)$ is either large or small.

If $P \in \mathcal{S}$, then $\mathrm{ind}(P) \leq 1$.

Let $P \notin \mathcal{S}$, $\mathrm{ind}(P) \leq q - 2$, $|\mathcal{S}| = 2q + \beta \leq 4q - 4$. Choose ℓ_∞ so

that $P \in \ell_\infty$, $|\ell_\infty \cap \mathcal{S}| = s \geq 2$, $(\infty) \notin S$, and $P \neq (\infty)$.
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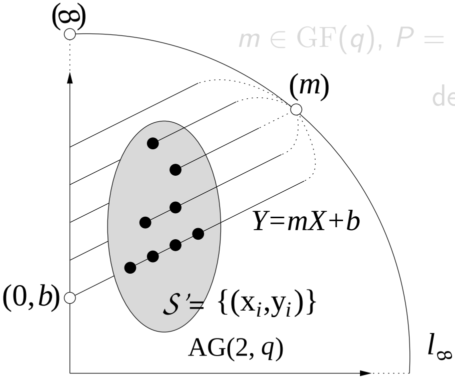
The Rédei polynomial

$$R(M,B) = \prod_{i=1}^{|\mathcal{S}\cap \mathrm{AG}(2,q)|} (Mx_i + B - y_i) \in \mathrm{GF}(q)[M,B]$$

$|\mathcal{S}' \cap \{Y = mX + b\}|$ = the multiplicity of the root b in $R(m,B)$



$m \in \mathrm{GF}(q)$, $P = (m) \notin \mathcal{S}$, ℓ_∞ a „standard" line

$\deg \gcd(R(m,B),(B^q - B)^2) =$

$0 \cdot$ #skew lines on (m)

$+1 \cdot$ #tangent lines on (m)

$+2 \cdot$ #standard lines on (m)

$= 2q - \mathrm{ind}(m)$

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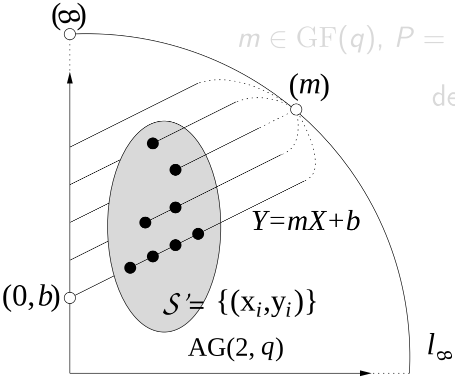

$m \in \mathrm{GF}(q)$, $P = (m) \notin \mathcal{S}$, ℓ_∞ a „standard" line

$\deg \gcd(R(m,B), (B^q - B)^2) =$

$0 \cdot$ #skew lines on (m)

$+1 \cdot$ #tangent lines on (m)

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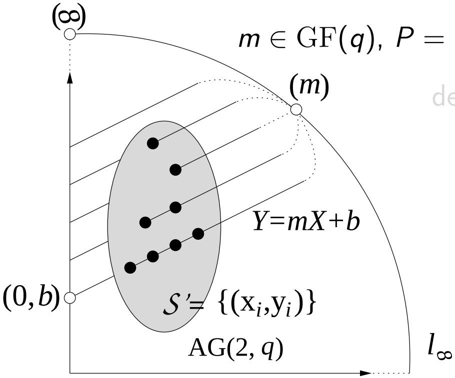

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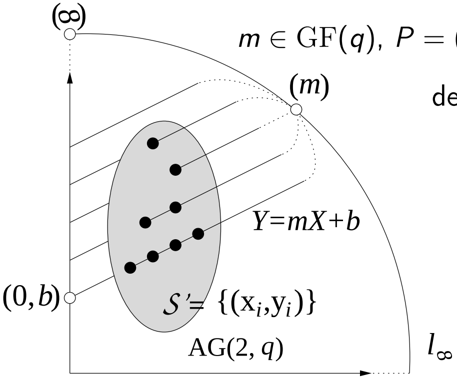

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For $z \in \mathbb{Z}$, $z^+ = \max\{z, 0\}$.

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Let $u, v \in \mathrm{GF}(q)[X, Y]$. Suppose that the coefficient of $X^{\deg(u)}$ in $u(X, Y)$ is not zero. For $y \in \mathrm{GF}(q)$, let

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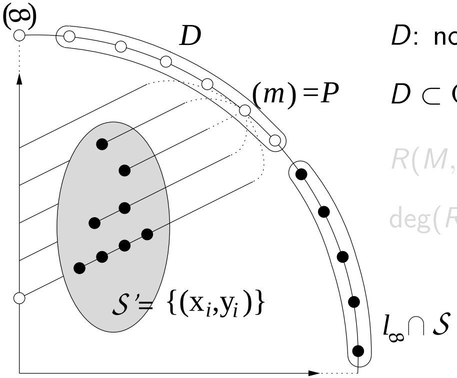

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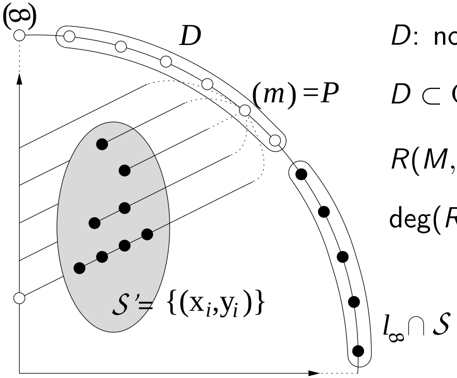

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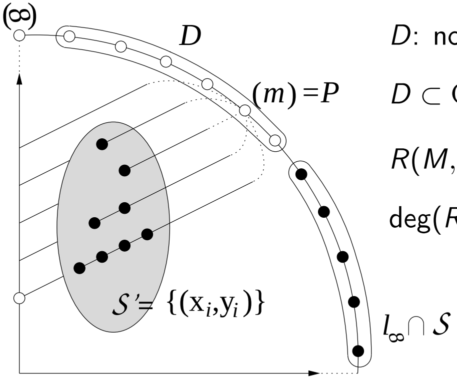

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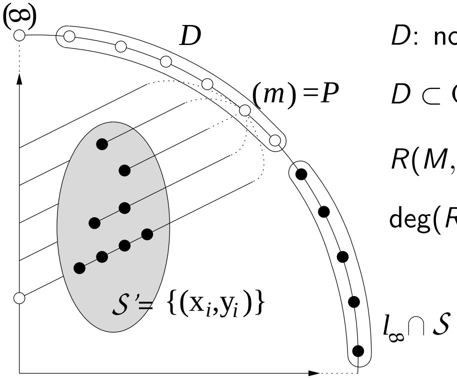

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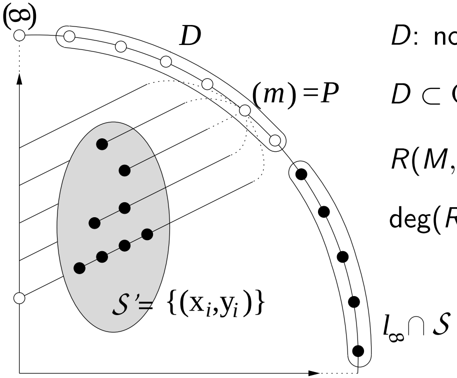

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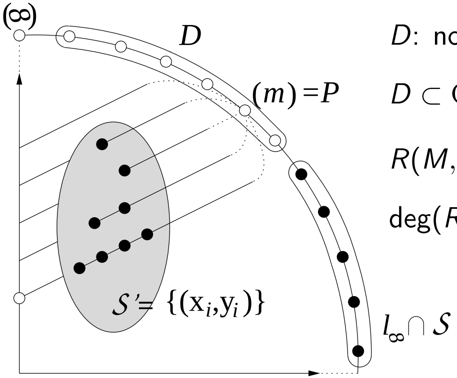

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There are no medium indices

Proposition

Let $P \notin \mathcal{S}$, $\beta \leq q/4 - 5/2$. Then $\operatorname{ind}(P) \leq 2$ or $\operatorname{ind}(P) \geq q - \beta - 2$.

Proof.

Recall that we have

$$\operatorname{ind}(P)^2 - (q - \beta)\operatorname{ind}(P) + 2q + \beta \geq 0.$$

Substituting $\operatorname{ind}(P) = 3$ or $\operatorname{ind}(P) = q - \beta - 3$, we get $\beta \geq (q - 9)/4$, a contradiction. □

Thus if $\mathcal{S}$ is not too large, then every point has a small or a large index.

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Points with large index block the non-standard lines

$\mathcal{T}$ is the set of points with large index.

Proposition

Assume $\beta < q/4 - 5/2$ and $q \geq 4$. If ℓ is tangent to $\mathcal{S}$, then $|\ell \cap \mathcal{T}| \geq 1$; if ℓ is skew to $\mathcal{S}$, then $|\ell \cap \mathcal{T}| \geq 2$.

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Proof.

Let ℓ be skew line; suppose that there is at most one point with large index on ℓ. Then there are at most q tangents; so $\delta \leq q + 2$. Let $P \in \ell$, $\mathrm{ind}(P) = 2$; then

$$0 \leq \mathrm{ind}(P)^2 - (q - \beta)\mathrm{ind}(P) + \delta \leq$$

$$4 - (q - \beta) \cdot 2 + 2 + q, \text{ so}$$

$$\beta \geq (q - 6)/2.$$

□

Points with large index block the non-standard lines

$\mathcal{T}$ is the set of points with large index.

Proposition

Assume $\beta \leq q/4 - 5/2$ and $q \geq 4$. If ℓ is tangent to $\mathcal{S}$, then $|\ell \cap \mathcal{T}| \geq 1$; if ℓ is skew to $\mathcal{S}$, then $|\ell \cap \mathcal{T}| \geq 2$.

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This means that $\mathcal{S} \cup \mathcal{T}$ is a double blocking set.

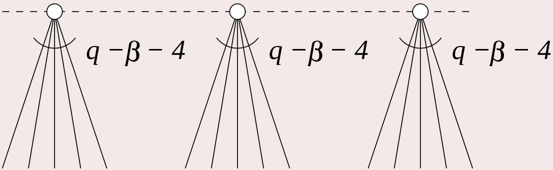
There are at most two points with large index

Proposition

Let $|\mathcal{S}| < 9q/4 - 3$ (that is, $\beta < q/4 - 3$). Then $|\mathcal{T}| \le 2$.

Proof.

Suppose that there are three points with index $\ge q - \beta - 2$. Then the number of tangents is at least $3(q - \beta - 4)$:



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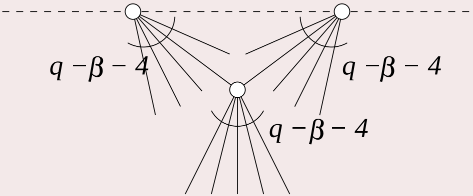

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Proof.

Suppose that there are three points with index $\geq q - \beta - 2$. Then the number of tangents is at least $3(q - \beta - 4)$. Thus

$$3q - 3\beta - 12 \leq |\mathcal{S}| = 2q + \beta,$$

whence $\beta \geq q/4 - 3$, a contradiction. □

Thank you for your attention!