Solving Binary Tomography from Morphological Skeleton via Optimization

Norbert Hantos, Péter Balázs, Kálmán Palágyi

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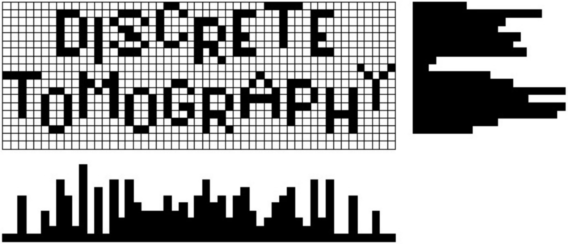
Motivation

Binary Tomography recreate binary images of cross-sections of objects from their projections

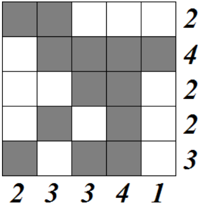


Extremely ambiguous if only a few projections are available
→ further information is needed

Image and projections

Image binary square matrix, $F_{n \times n}$

Projections sum of rows and columns, $\mathcal{H}(F)$, $\mathcal{V}(F)$



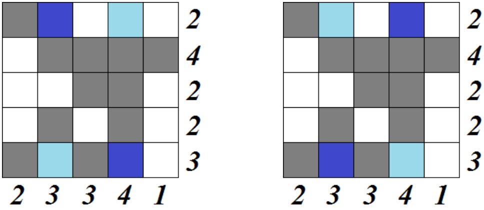
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\mathcal{H}_i(F) = \sum_{j=1}^{n} F_{ij} \qquad \mathcal{V}_j(F) = \sum_{i=1}^{n} F_{ij}$$

Switching components

- Submatrix of an image in size of 2×2 where switching 0-s and 1-s do not change the projections
- Necessary and sufficent condition for ambiguity

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



Morphological skeleton

- Representation of shapes
- Close to the topological skeleton
- Easy to generate
- Can store the image uniquely (with some additional information)

Morphological skeleton

The morphological skeleton of the binary image F with the structuring element Y can be extracted with morphological operators (erosion and dilation).

$$S(F,Y) = \bigcup_k S_k(F,Y),$$

where

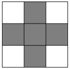
$$S_k(F,Y) = (F \ominus_k Y) \ \setminus \ \left[(F \ominus_{k+1} Y) \oplus Y\right].$$

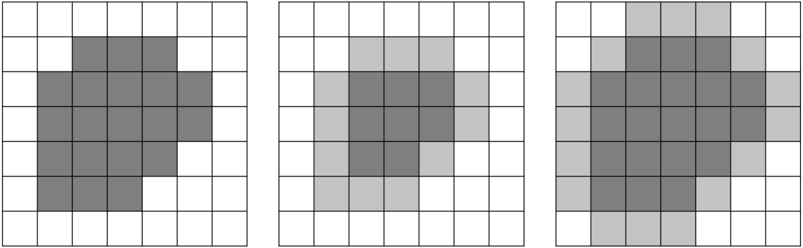
Morphological skeleton

Example for erosion and dilation

From now let the structuring element

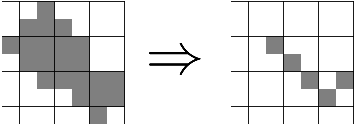
$$Y := \{(-1,0), (0,-1), (0,0), (0,1), (1,0)\}.$$

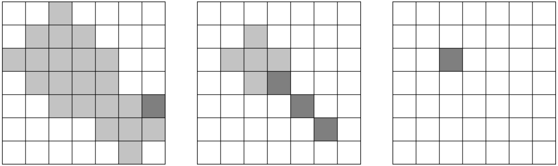




Original F Erosion $F \ominus Y$ Dilation $F \oplus Y$

Morphological skeleton





$k = 0$ $k = 1$ $k = 2$

Non-white pixels indicate $(F \ominus_k Y)$. Light grays indicate $\left[(F \ominus_{k+1} Y) \oplus Y\right]$, dark grays are the difference (i.e., $S_k(F, Y)$).

Morphological skeleton

If we know the structuring element Y and the $S_k(F,Y)$ for each k, then we can reconstruct the orignal image:

$$F = \bigcup_k \left[S_k(F,Y) \oplus_k Y \right],$$

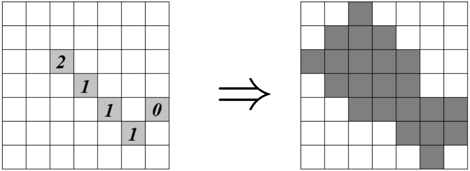
or in a different form:

$$F = \bigcup_{p \in S(F,Y)} (p \oplus_{k_p} Y),$$

where k_p is a unique value for every p such that $p \in S_{k_p}(F)$.

Morphological skeleton

Let $K(S) := (k_{p_1}, k_{p_2}, ..., k_{p_{|S|}})$ the series of the k_p values, $p_i \in S = S(F, Y)$, and F is uniquely determined by $K(S)$ and Y.



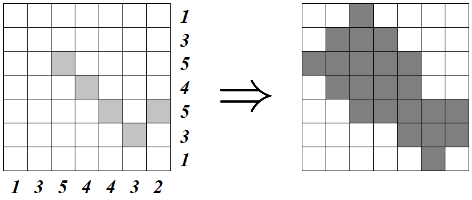
S and the corresponding $K(S)$ (left), the recreated F (right).

Reconstruction with morphological skeleton

Main task

Given vectors H and V, morphological skeleton S and structuring element Y. We want to find $K(S)$ in a way that the corresponding F is the closest to the required projections:

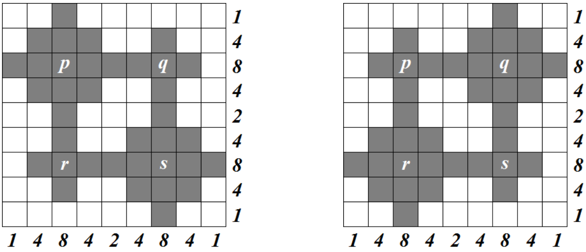
$$f\big(K(S)\big) = ||H - \mathcal{H}(F)||_2 + ||V - \mathcal{V}(F)||_2 \;\rightarrow\; \min$$



Theoretical results

Theorem (Un-uniqueness)

The skeleton based reconstruction is not unique, since there could be an image pair F_1 and F_2 that they have the same projections and skeleton, however $F_1 \neq F_2$.



Note that $(2,1,1,2) = K_1(S) \quad \neq \quad K_2(S) = (1,2,2,1)$.

Theoretical results

Given H, V projection vectors, S skeletal set. Does any binary image F exist where $H = \mathcal{H}(F)$, $V = \mathcal{V}(F)$, $S = S(F, Y)$ and F is 4-connected?

Theorem (NP-completedness)

The problem above is NP-complete.

Note that we fixed the structuring element Y.

Conjecture

The problem above is still NP-complete even without requiring the 4-connectedness.

Theoretical results

Theorem (Skeletal smoothness)

For any image F and any skeletal points $p, q \in S(F, Y)$,

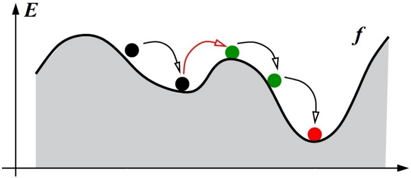
$$|k_p - k_q| < ||p - q||_1,$$

where $||.||_1$ denotes the Manhattan norm.

As a special case, if p is 8-adjacent to q, then $|k_p - k_q| < 2$, that we are to use during the reconstruction.

Simulated Annealing

- Iterative stochastic method for finding a global minimum of a function
- Could find a near-optimal minimum in a reasonable time
- Has many technical parameters, in our case:
  - Variables: $K(S)$
  - Energy function: $f\big(K(S)\big) \rightarrow \min$
  - Stopping criteria: iteration number M or zero energy
  - Annealing schedule: $T(t) = T_0 \cdot \left(\frac{T_s}{T_0}\right)^{\frac{t}{M}}$, where t denotes time



Simulated Annealing
NVC

NVC *(No Vase Constraint)* model:

- $f\big(K(S)\big) = ||H - \mathcal{H}(F)||_2 + ||V - \mathcal{V}(F)||_2$
- Changing a variable: simply change an element $k_p \in K(S)$ randomly

Simulated Annealing
DVC

DVC *(Dynamic Vase Constraint)* model:

- f is the same as in NVC
- Changing a variable: change an element $k_p \in K(S)$ such that $|k_p - k_q| < C(t)$ for each q 8-adjacent to p and

$$C(t) = \left\lceil C_0 \cdot \left(\frac{C_s}{C_0}\right)^{\frac{t}{M}} \right\rceil$$

Note that $C(t)$ is monotonically decreasing and the limit is 1.

Simulated Annealing
CEF

CEF *(Combined Energy Function)* model:

- $$f\big(K(S)\big) = \alpha \cdot \Big(||H - \mathcal{H}(F)||_2 + ||V - \mathcal{V}(F)||_2\Big) + \\ + (1-\alpha) \cdot \sum_{||p-q||_1 \leq 1} h(k_p, k_q) \ ,$$

  where

  $$h(k_p, k_q) = \begin{cases} 0 & \text{if } |k_p - k_q| \leq 1 \\ |k_p - k_q|/2 & \text{otherwise.} \end{cases}$$

- Changing a variable: the same as in NVC

Results

- Artificial images in size of 256×256
  1. Simple convex shape
  2. Grid of comvex shapes
  3. Random set of convex shapes
  4. Miscellaneous images
- Technical parameters: $M = 50000$, $T_0 = 10$, $T_s = 0.001$
- Average of 5 runs
- Testing environment: Intel Core 2 Duo T250, 1.5 GHz, 2GB RAM

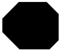
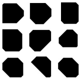
Error measurement

$$E = \sqrt{\sum_{i=1}^{2n} (b_i - b'_i)^2} ,$$

where b_i and b'_i are the elements of the original and the reconstructed projections, respectively.

Results

Image	Method	CPU (ms)	E
	NVC	3842	1060
	DVC_{10}	4030	98
	DVC_{5}	4116	97
	**DVC$_1$**	**4563**	**18**
	$\mathrm{CEF}_{0.3}$	4358	2468
	$\mathrm{CEF}_{0.5}$	4415	1675
	$\mathrm{CEF}_{0.7}$	4435	1305
	NVC	3784	3405
	DVC$_{10}$	**3038**	**1291**
	DVC_{5}	3164	4288
	DVC_{1}	3566	5307
	$\mathrm{CEF}_{0.3}$	5412	5665
	$\mathrm{CEF}_{0.5}$	5387	4829
	$\mathrm{CEF}_{0.7}$	5328	3212

Image	Method	CPU (ms)	E
	NVC	7276	1285
	DVC_{10}	7900	174
	DVC_{5}	8127	146
	**DVC$_1$**	**4473**	**0**
	$\mathrm{CEF}_{0.3}$	7626	2578
	$\mathrm{CEF}_{0.5}$	7665	1849
	$\mathrm{CEF}_{0.7}$	7691	1505
	NVC	**4346**	**6136**
	DVC_{10}	4733	1066145
	DVC_{5}	4609	1722350
	DVC_{1}	4926	3302481
	$\mathrm{CEF}_{0.3}$	7308	14371
	$\mathrm{CEF}_{0.5}$	7243	8896
	$\mathrm{CEF}_{0.7}$	7222	7402

Results

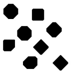
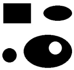
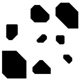
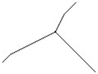
Image	Method	CPU (ms)	E
	NVC	1666	1341
	DVC_{10}	**1215**	**292**
	DVC_5	1234	314
	DVC_1	1302	294
	$CEF_{0.3}$	2904	2534
	$CEF_{0.5}$	2827	1950
	$CEF_{0.7}$	2851	1732
	**NVC**	**3537**	**2530**
	DVC_{10}	2852	9154
	DVC_5	2981	13138
	DVC_1	3226	67493
	$CEF_{0.3}$	6380	5183
	$CEF_{0.5}$	6367	4102
	$CEF_{0.7}$	6343	3029

Image	Method	CPU (ms)	E
	**NVC**	**2165**	**2709**
	DVC_{10}	1713	6042
	DVC_5	1724	7962
	DVC_1	1910	6360
	$CEF_{0.3}$	4123	5688
	$CEF_{0.5}$	4131	4178
	$CEF_{0.7}$	4114	3346
	**NVC**	**2757**	**4034**
	DVC_{10}	2304	4523
	DVC_5	2467	7472
	DVC_1	2430	13096
	$CEF_{0.3}$	8884	6663
	$CEF_{0.5}$	8856	5012
	$CEF_{0.7}$	8959	4407

Results

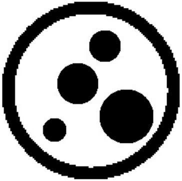
Examples

Original

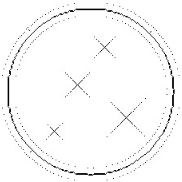


Skeleton

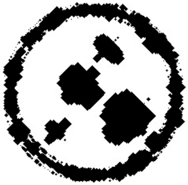
Result with $\text{CEF}_{0.5}$



Original



Skeleton



Result with NVC

Conclusions

- Image reconstruction is extremely underdetermined if only a few projections are used
- Morphological skeleton can reduce the ambiguity, however, the reconstruction problem is (possibly) NP-complete
- 3 variants of SA are tested on artifical images
  - **NVC** generally acceptable reconstruction
  - **DVC** smoother results, sometimes converges very slowly (highly depends on the initial image)
  - **CEF** similar results as NVC, computationally intensive

Future work

- Prove NP- (or P-) completedness of the original task and its variants (such as h-convex images)
- Examine strategies for choosing the initial image for SA
- Find a more sophisticated function minimizer
- Try other prior information, such as smoothness on the boundary
- Study the robustness of the reconstruction when the projections are corrupted by noise

Thank you for your attention!

- Hantos, N., Balázs, P., Palágyi, K.: *Binary image reconstruction from two projections and skeletal information*. 15th International Workshop on Combinatorial Image Analysis (IWCIA 2012), LNCS 7655, Springer, Heidelberg, 263–274 (2012).
- Hantos, N., Balázs, P.: *The reconstruction of polyominoes from horizontal and vertical projections and morphological skeleton is NP-complete*. Accepted for publication in Fundamenta Informaticae (2012).
- Herman, G.T., Kuba, A. (eds.): *Advances in Discrete Tomography and Its Applications*. Birkhäuser, Boston (2007).
- Gonzalez, R.C., Woods, R.E.: *Digital Image Processing (3rd Edition)*. Prentice Hall (2008).
- Kirkpatrick, S., Gelatt Jr., C.D., Vecchi, M.P.: *Optimization by simulated annealing*. Science 220, 671–680 (1983).

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