Solving Binary Tomography from Morphological Skeleton via Optimization

Norbert Hantos, Péter Balázs, Kálmán Palágyi

Veszprém, December 11-14, 2012

VOCAL 2012
Contents

1 Introduction
   Motivation
   Image and projections
   Switching components
   Morphological skeleton

2 Reconstruction with morphological skeleton
   Main task
   Theoretical results
   Simulated Annealing

3 Results
Binary Tomography recreate binary images of cross-sections of objects from their projections.

Extremely ambiguous if only a few projections are available
→ further information is needed
Image and projections

**Image** binary square matrix, $F_{n \times n}$

**Projections** sum of rows and columns, $\mathcal{H}(F)$, $\mathcal{V}(F)$

\[
\mathcal{H}_i(F) = \sum_{j=1}^{n} F_{ij}
\]

\[
\mathcal{V}_j(F) = \sum_{i=1}^{n} F_{ij}
\]
Switching components

- Submatrix of an image in size of $2 \times 2$ where switching 0-s and 1-s do not change the projections
- Necessary and sufficient condition for ambiguity

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix} \iff \begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

```
\begin{array}{|c|c|c|c|}
\hline
  & 2 & 3 & 4 \\
\hline
2 & & & \\
3 & & & \\
4 & & & \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
  & 2 & 3 & 4 \\
\hline
2 & & & \\
3 & & & \\
4 & & & \\
\hline
\end{array}
```
Morphological skeleton

- Representation of shapes
- Close to the topological skeleton
- Easy to generate
- Can store the image uniquely (with some additional information)
Morphological skeleton

The morphological skeleton of the binary image $F$ with the structuring element $Y$ can be extracted with morphological operators (erosion and dilation).

$$S(F, Y) = \bigcup_{k} S_k(F, Y),$$

where

$$S_k(F, Y) = (F \ominus_k Y) \setminus [(F \ominus_{k+1} Y) \oplus Y].$$
Morphological skeleton

Example for erosion and dilation

From now let the structuring element

\[ Y := \{(-1, 0), (0, -1), (0, 0), (0, 1), (1, 0)\}. \]

Original \( F \)  \hspace{1cm} Erosion \( F \ominus Y \)  \hspace{1cm} Dilation \( F \oplus Y \)
Non-white pixels indicate \((F \ominus_k Y)\). Light grays indicate \([ (F \ominus_{k+1} Y) \oplus Y ]\), dark grays are the difference (i.e., \(S_k(F,Y)\)).
Morphological skeleton

If we know the structuring element $Y$ and the $S_k(F,Y)$ for each $k$, then we can reconstruct the original image:

$$F = \bigcup_k \left[ S_k(F,Y) \oplus_k Y \right],$$

or in a different form:

$$F = \bigcup_{p \in S(F,Y)} (p \oplus_{k_p} Y),$$

where $k_p$ is a unique value for every $p$ such that $p \in S_{k_p}(F)$. 

Norbert Hantos, Péter Balázs, Kálmán Palágyi

Binary Tomography from Morphological Skeleton
Let \( K(S) := (k_{p_1}, k_{p_2}, \ldots, k_{p_{|S|}}) \) the series of the \( k_p \) values, \( p_i \in S = S(F,Y) \), and \( F \) is uniquely determined by \( K(S) \) and \( Y \).

\[
\begin{array}{c}
\begin{array}{cccc}
2 & & & \\
1 & 1 & 1 & \\
0 & & & \\
\end{array}
\end{array} \quad \Rightarrow \quad
\begin{array}{c}
\begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
\end{array}
\end{array}
\]

\( S \) and the corresponding \( K(S) \) (left), the recreated \( F \) (right).
Main task

Given vectors $H$ and $V$, morphological skeleton $S$ and structuring element $Y$. We want to find $K(S)$ in a way that the corresponding $F$ is the closest to the required projections:

$$f(K(S)) = \|H - \mathcal{H}(F)\|_2 + \|V - \mathcal{V}(F)\|_2 \rightarrow \min$$
Theoretical results

**Theorem (Un-uniqueness)**

The skeleton based reconstruction is not unique, since there could be an image pair $F_1$ and $F_2$ that they have the same projections and skeleton, however $F_1 \neq F_2$.

Note that $(2, 1, 1, 2) = K_1(S) \neq K_2(S) = (1, 2, 2, 1)$. 
Theoretical results

Given $H, V$ projection vectors, $S$ skeletal set. Does any binary image $F$ exist where $H = \mathcal{H}(F)$, $V = \mathcal{V}(F)$, $S = S(F,Y)$ and $F$ is 4-connected?

**Theorem (NP-completedness)**

The problem above is NP-complete.

Note that we fixed the structuring element $Y$.

**Conjecture**

The problem above is still NP-complete even without requiring the 4-connectedness.
Theoretical results

Theorem (Skeletal smoothness)

For any image $F$ and any skeletal points $p, q \in S(F, Y)$,

$$|k_p - k_q| < ||p - q||_1,$$

where $||.||_1$ denotes the Manhattan norm.

As a special case, if $p$ is 8-adjacent to $q$, then $|k_p - k_q| < 2$, that we are to use during the reconstruction.
Simulated Annealing

- Iterative stochastic method for finding a global minimum of a function
- Could find a near-optimal minimum in a reasonable time
- Has many technical parameters, in our case:
  - Variables: \( K(S) \)
  - Energy function: \( f(K(S)) \rightarrow \min \)
  - Stopping criteria: iteration number \( M \) or zero energy
  - Annealing schedule: \( T(t) = T_0 \cdot \left( \frac{T_s}{T_0} \right)^{\frac{t}{M}} \), where \( t \) denotes time
Simulated Annealing

NVC (No Vase Constraint) model:

- \[ f(K(S)) = \| H - \mathcal{H}(F) \|_2 + \| V - \mathcal{V}(F) \|_2 \]
- Changing a variable: simply change an element \( k_p \in K(S) \) randomly
Simulated Annealing

DVC

DVC (Dynamic Vase Constraint) model:

• $f$ is the same as in NVC

• Changing a variable: change an element $k_p \in K(S)$ such that $|k_p - k_q| < C(t)$ for each $q$ 8-adjacent to $p$ and

$$C(t) = \left\lfloor C_0 \cdot \left( \frac{C_s}{C_0} \right)^{\frac{t}{M}} \right\rfloor$$

Note that $C(t)$ is monotonically decreasing and the limit is 1.
Simulated Annealing

CEF

CEF (Combined Energy Function) model:

\[ f(K(S)) = \alpha \cdot \left( \| H - H(F) \|_2 + \| V - V(F) \|_2 \right) + \\
+ (1 - \alpha) \cdot \sum_{||p-q||_1 \leq 1} h(k_p, k_q), \]

where

\[ h(k_p, k_q) = \begin{cases} 
0 & \text{if } |k_p - k_q| \leq 1 \\
|k_p - k_q|/2 & \text{otherwise.}
\end{cases} \]

- Changing a variable: the same as in NVC
Results

- Artificial images in size of $256 \times 256$
  1. Simple convex shape
  2. Grid of convex shapes
  3. Random set of convex shapes
  4. Miscellaneous images
- Technical parameters: $M = 50000$, $T_0 = 10$, $T_s = 0.001$
- Average of 5 runs
- Testing environment: Intel Core 2 Duo T250, 1.5 GHz, 2GB RAM

Error measurement

$$E = \sqrt{\frac{2n}{\sum_{i=1}^{2n} (b_i - b_i')^2}}$$

where $b_i$ and $b_i'$ are the elements of the original and the reconstructed projections, respectively.
## Results

<table>
<thead>
<tr>
<th>Image</th>
<th>Method</th>
<th>CPU (ms)</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Triangle" /></td>
<td>NVC</td>
<td>3842</td>
<td>1060</td>
</tr>
<tr>
<td></td>
<td>DVC$_{10}$</td>
<td>4030</td>
<td>98</td>
</tr>
<tr>
<td></td>
<td>DVC$_{5}$</td>
<td>4116</td>
<td>97</td>
</tr>
<tr>
<td></td>
<td>DVC$_1$</td>
<td>4563</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.3}$</td>
<td>4358</td>
<td>2468</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.5}$</td>
<td>4415</td>
<td>1675</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.7}$</td>
<td>4435</td>
<td>1305</td>
</tr>
<tr>
<td><img src="image2.png" alt="Octagon" /></td>
<td>NVC</td>
<td>3784</td>
<td>3405</td>
</tr>
<tr>
<td></td>
<td>DVC$_{10}$</td>
<td>3038</td>
<td>1291</td>
</tr>
<tr>
<td></td>
<td>DVC$_{5}$</td>
<td>3164</td>
<td>4288</td>
</tr>
<tr>
<td></td>
<td>DVC$_1$</td>
<td>3566</td>
<td>5307</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.3}$</td>
<td>5412</td>
<td>5665</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.5}$</td>
<td>5387</td>
<td>4829</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.7}$</td>
<td>5328</td>
<td>3212</td>
</tr>
<tr>
<td><img src="image3.png" alt="Four Octagons" /></td>
<td>NVC</td>
<td>3784</td>
<td>3405</td>
</tr>
<tr>
<td></td>
<td>DVC$_{10}$</td>
<td>3038</td>
<td>1291</td>
</tr>
<tr>
<td></td>
<td>DVC$_{5}$</td>
<td>3164</td>
<td>4288</td>
</tr>
<tr>
<td></td>
<td>DVC$_1$</td>
<td>3566</td>
<td>5307</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.3}$</td>
<td>5412</td>
<td>5665</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.5}$</td>
<td>5387</td>
<td>4829</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.7}$</td>
<td>5328</td>
<td>3212</td>
</tr>
<tr>
<td><img src="image4.png" alt="Twelve Octagons" /></td>
<td>NVC</td>
<td>4346</td>
<td>6136</td>
</tr>
<tr>
<td></td>
<td>DVC$_{10}$</td>
<td>4733</td>
<td>1066145</td>
</tr>
<tr>
<td></td>
<td>DVC$_{5}$</td>
<td>4609</td>
<td>1722350</td>
</tr>
<tr>
<td></td>
<td>DVC$_1$</td>
<td>4926</td>
<td>3302481</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.3}$</td>
<td>7308</td>
<td>14371</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.5}$</td>
<td>7243</td>
<td>8896</td>
</tr>
<tr>
<td></td>
<td>CEF$_{0.7}$</td>
<td>7222</td>
<td>7402</td>
</tr>
</tbody>
</table>
## Results

<table>
<thead>
<tr>
<th>Image</th>
<th>Method</th>
<th>CPU (ms)</th>
<th>(E)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td>NVC</td>
<td>1215</td>
<td>292</td>
</tr>
<tr>
<td></td>
<td>DVC(_{10})</td>
<td>1234</td>
<td>314</td>
</tr>
<tr>
<td></td>
<td>DVC(_{5})</td>
<td>1302</td>
<td>294</td>
</tr>
<tr>
<td></td>
<td>DVC(_{1})</td>
<td>2904</td>
<td>2534</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.3})</td>
<td>2827</td>
<td>1950</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.5})</td>
<td>2851</td>
<td>1732</td>
</tr>
<tr>
<td><img src="image2.png" alt="Image" /></td>
<td>NVC</td>
<td>3537</td>
<td>2530</td>
</tr>
<tr>
<td></td>
<td>DVC(_{10})</td>
<td>2852</td>
<td>9154</td>
</tr>
<tr>
<td></td>
<td>DVC(_{5})</td>
<td>2981</td>
<td>13138</td>
</tr>
<tr>
<td></td>
<td>DVC(_{1})</td>
<td>3226</td>
<td>67493</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.3})</td>
<td>6380</td>
<td>5183</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.5})</td>
<td>6367</td>
<td>4102</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.7})</td>
<td>6343</td>
<td>3029</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Image</th>
<th>Method</th>
<th>CPU (ms)</th>
<th>(E)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image3.png" alt="Image" /></td>
<td>NVC</td>
<td>2165</td>
<td>2709</td>
</tr>
<tr>
<td></td>
<td>DVC(_{10})</td>
<td>1713</td>
<td>6042</td>
</tr>
<tr>
<td></td>
<td>DVC(_{5})</td>
<td>1724</td>
<td>7962</td>
</tr>
<tr>
<td></td>
<td>DVC(_{1})</td>
<td>1910</td>
<td>6360</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.3})</td>
<td>4123</td>
<td>5688</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.5})</td>
<td>4131</td>
<td>4178</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.7})</td>
<td>4114</td>
<td>3346</td>
</tr>
<tr>
<td><img src="image4.png" alt="Image" /></td>
<td>NVC</td>
<td>2757</td>
<td>4034</td>
</tr>
<tr>
<td></td>
<td>DVC(_{10})</td>
<td>2304</td>
<td>4523</td>
</tr>
<tr>
<td></td>
<td>DVC(_{5})</td>
<td>2467</td>
<td>7472</td>
</tr>
<tr>
<td></td>
<td>DVC(_{1})</td>
<td>2430</td>
<td>13096</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.3})</td>
<td>8884</td>
<td>6663</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.5})</td>
<td>8856</td>
<td>5012</td>
</tr>
<tr>
<td></td>
<td>CEF(_{0.7})</td>
<td>8959</td>
<td>4407</td>
</tr>
</tbody>
</table>
Results

Examples

Original
Skeleton
Result with $CEF_{0.5}$

Original
Skeleton
Result with NVC
• Image reconstruction is extremely underdetermined if only a few projections are used
• Morphological skeleton can reduce the ambiguity, however, the reconstruction problem is (possibly) NP-complete
• 3 variants of SA are tested on artificial images
  - NVC generally acceptable reconstruction
  - DVC smoother results, sometimes converges very slowly (highly depends on the initial image)
  - CEF similar results as NVC, computationally intensive
Future work

- Prove NP- (or P-) completedness of the original task and its variants (such as $h$-convex images)
- Examine strategies for choosing the initial image for SA
- Find a more sophisticated function minimizer
- Try other prior information, such as smoothness on the boundary
- Study the robustness of the reconstruction when the projections are corrupted by noise
Thank you for your attention!


Hantos, N., Balázs, P.: *The reconstruction of polyominoes from horizontal and vertical projections and morphological skeleton is NP-complete*. Accepted for publication in Fundamenta Informaticae (2012).


The presentation is supported by the European Union and co-funded by the European Social Fund. 
Project title: “Broadening the knowledge base and supporting the long term professional sustainability of the Research University Centre of Excellence at the University of Szeged by ensuring the rising generation of excellent scientists.”

Project number: TÁMOP-4.2.2/B-10/1-2010-0012