An Energy Minimization Reconstruction Algorithm for Multivalued Discrete Tomography

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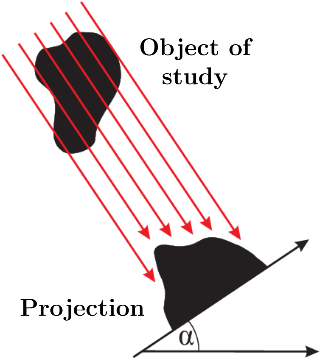
Contents

Transmission tomography

- We are interested in the inner structure of some given object.
- We can measure the projections of the object of study (the densities of the object on the path of some projection beams).
- The goal is to reconstruct the original structure from a given set of projections.



Transmission tomography

- The object of study is represented by a function $f(u, v)$.

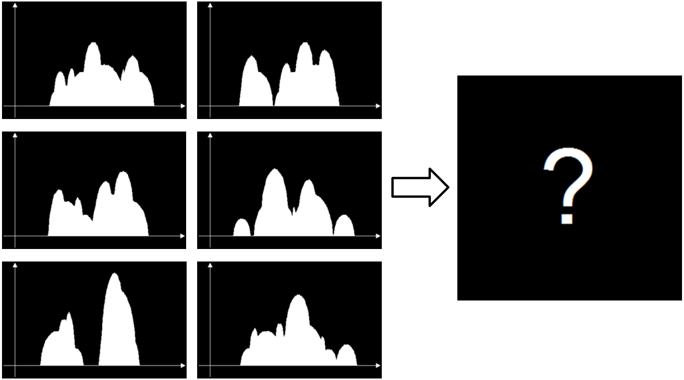
$$f : \mathbb{R}^2 \to \mathbb{R} \tag{1}$$

- We take the line integrals of the image (Radon-Transform).

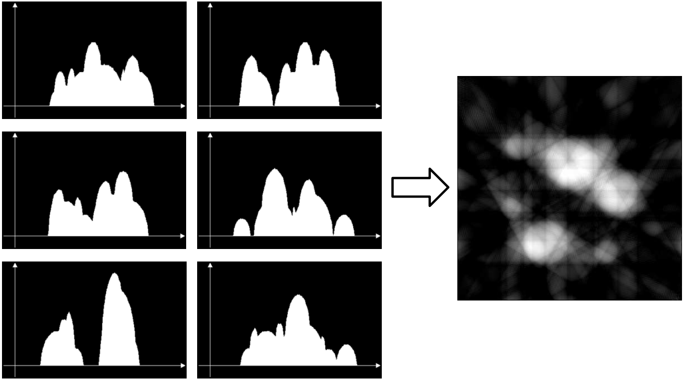
$$[\mathcal{R}f](\alpha, t) = \int_{-\infty}^{\infty} f(t\cos(\alpha) - q\sin(\alpha), t\sin(\alpha) + q\cos(\alpha))\ dq \tag{2}$$

- We are looking for an $f'(u, v)$ function that has the same projections as the original $f(u, v)$.

Transmission tomography



Transmission tomography



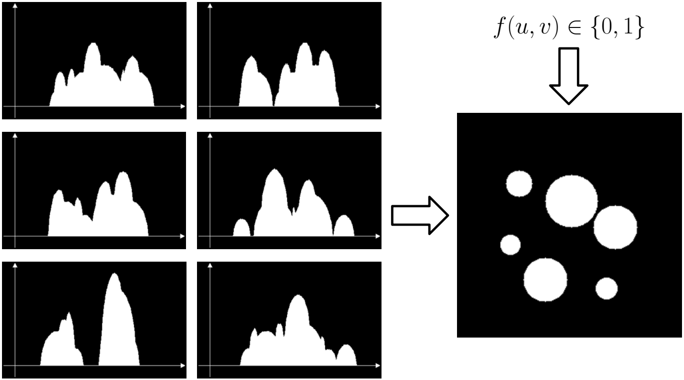
Discrete Tomography

In discrete tomography we assume that the object of study consists of only a few known materials.

$$f(u, v) \in \{l_1, l_2, \ldots, l_c\} \quad (3)$$

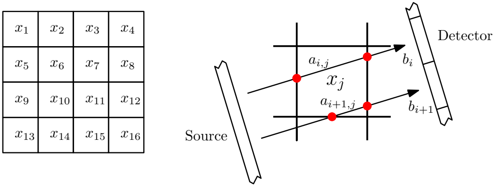
With this information we can gain accurate reconstructions from only few (say, 2-10) projections.

Discrete Tomography



Formulation of the reconstruction problem

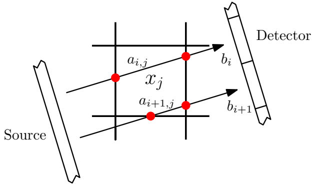
- We assume a discrete representation of the object of study (i.e., it is represented on an $n \times n$ sized discrete image).
- The projections are given by the integrals of the image along a set of straight lines.



Formulation of the reconstruction problem

- With this the reconstruction problem can be reformulated a a system of equations $\mathbf{Ax} = \mathbf{b}$, where:
  - $\mathbf{b}$, is the vector of m projection values,
  - $\mathbf{x}$, represents the vector of the image pixel values,
  - $\mathbf{A}$, describes the connection between the image pixels, and the projection values, with all a_{ij} giving the length line segment of the i-th projection line in the j pixel.
- We will further assume, that the pixel intensities are elements of a predefined set $L = \{l_1, l_2, \ldots, l_c\}$.

x_1	x_2	x_3	x_4
x_5	x_6	x_7	x_8
x_9	x_{10}	x_{11}	x_{12}
x_{13}	x_{14}	x_{15}	x_{16}



Energy function

With the algebraic formulation of the reconstruction algorithm, one can construct an energy function that takes its minima in the correct reconstructions.

Projection correctness: (convex)
Minimal if the result satisfies projections.

Smoothness term: (convex)
Minimal, if result contains large homogeneous regions.

Discretizing term: (non-convex)
Minimal at discrete solutions.

$$\mathcal{E}_\mu(\mathbf{x}) := \overbrace{\frac{1}{2} \cdot \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2}^{} + \overbrace{\frac{\alpha}{2} \cdot \sum_{i=1}^{n^2} \sum_{j \in N_4(i)} (x_i - x_j)^2}^{} + \overbrace{\mu \cdot g(\mathbf{x})}^{}, \quad \mathbf{x} \in [l_0, l_c]^{n^2}$$

Discretizing term

The part of the energy function responsible for the discretizing can be given in the form

$$g(\mathbf{x}) = \sum_{i=1}^{n^2} g_p(x_i)\ , \quad i \in \{1, 2, \ldots, n^2\}\ , \tag{4}$$

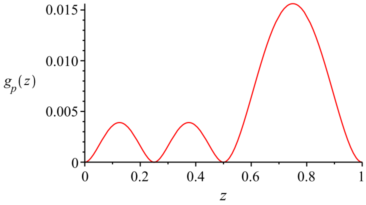
where

$$g_p(z) = \begin{cases} \frac{\left[(z-l_{j-1})\cdot(z-l_j)\right]^2}{2\cdot(l_j-l_{j-1})^2}, & \text{ha } z \in [l_{j-1}, l_j] \text{ for each } j \in \{2, \ldots, c\}, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Discretizing term

The discretizing term is given as the sum of one-dimensional discretizing functions written on each pixel value, which

- takes a 0 minimum, at the discrete values of L,
- and takes high positive values between the desired intensities.



Example of the discretizing function of one pixel with $L = \{0, 0.25, 0.5, 1\}$ expected intensities.

Basic process of the optimization

- The minimized energy function is basically constructed of two parts:
  - Two convex terms responsible for projection correctness, and "smoothness".
  - A non-convex discretizing term preferring discrete solutions of L.

$$\mathcal{E}_{\mu}(\mathbf{x}) := \frac{1}{2} \cdot \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \frac{\alpha}{2} \cdot \sum_{i=1}^{n^2} \sum_{j \in N_4(i)} (x_i - x_j)^2 + \mu \cdot g(\mathbf{x}), \quad \mathbf{x} \in [l_1, l_c]^{n^2} \quad (5)$$

Basic process of the optimization

- We assume that, the most important part of the energy function is the projection correctness term, and start the optimization with a gradient method as follows:
  1. At the beginning we omit discretization.
  2. We start running a gradient method from an initial solution.
  3. As the projections of the current intermediate solution get closer to the desired ones, we slowly start to increase the weight of the discretization
  4. When the iteration does not make significant changes of the results, we stop the process.

The algorithm

Input: $\mathbf{A}$ projection matrix, $\mathbf{b}$ expected projection values, $\mathbf{x}^0$ initial solution, $\alpha, \mu, \sigma \geq 0$ predefined constants, and L list of expected intensities.

```
λ ← an upper bound for the largest eigenvalue of the (AᵀA + α·S) matrix.
k ← 0
repeat
    v ← Aᵀ(Axᵏ − b).
    w ← Sxᵏ.
    for each i ∈ {1, 2, ..., n²} do
        (step 7)
        (step 8)
    end for
    k ← k + 1
until a stopping criterium is met.
Apply a discretization of xᵏ to gain fully discrete results.
```

Step 7:

$$y_i^{k+1} \leftarrow x_i^k - \frac{v_i + \alpha \cdot w_i + \mu \cdot G_{0,\sigma}(v_i) \cdot \nabla g_p(x_i^k)}{\lambda + \mu}$$

Step 8:

$$x_i^{k+1} \leftarrow \begin{cases} l_1, & \text{if } y_i^{k+1} < l_1, \\ y_i^{k+1}, & \text{if } l_1 \leq y_i^{k+1} \leq l_c, \\ l_c, & \text{if } l_c < y_i^{k+1}. \end{cases}$$

The algorithm

Input: $\mathbf{A}$ projection matrix, $\mathbf{b}$ expected projection values, $\mathbf{x}^0$ initial solution, $\alpha, \mu, \sigma \geq 0$ predefined constants, and L list of expected intensities.

$\lambda \leftarrow$ an upper bound for the largest eigenvalue of the $(\mathbf{A}^T\mathbf{A} + \alpha \cdot \mathbf{S})$ matrix.

$k \leftarrow 0$

repeat

  $\mathbf{v} \leftarrow \mathbf{A}^T(\mathbf{A}\mathbf{x}^k - \mathbf{b})$.

  $\mathbf{w} \leftarrow S\mathbf{x}^k$.

  for each $i \in \{1, 2, \ldots, n^2\}$ **do**

    $y_i^{k+1} \leftarrow x_i^k - \frac{v_i + \alpha \cdot w_i + \mu \cdot G_{0,\sigma}(v_i) \cdot \nabla g_p(x_i^k)}{\lambda + \mu}$

    $x_i^{k+1} \leftarrow \begin{cases} l_1, & \text{if } y_i^{k+1} < l_1, \\ y_i^{k+1}, & \text{if } l_1 \leq y_i^{k+1} \leq l_c, \\ l_c, & \text{if } l_c < y_i^{k+1}. \end{cases}$

  end for

  $k \leftarrow k + 1$

until a stopping criterium is met.

Apply a discretization of $\mathbf{x}^k$ to gain fully discrete results.

Description of the iteration

$$y_i^{k+1} \leftarrow x_i^k - \frac{v_i + \alpha \cdot w_i + \mu \cdot G_{0,\sigma}(v_i) \cdot \nabla g_p(x_i^k)}{\lambda + \mu}$$

Description of the iteration

$$y_i^{k+1} \longleftarrow x_i^k - \frac{v_i + \alpha \cdot w_i + \mu \cdot G_{0,\sigma}(v_i) \cdot \nabla g_p(x_i^k)}{\lambda + \mu}$$

previous iteration step

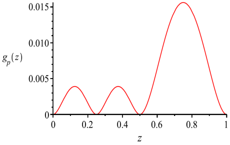
Description of the iteration

$$y_i^{k+1} \leftarrow x_i^k - \frac{v_i + \alpha \cdot w_i + \mu \cdot G_{0,\sigma}(v_i) \cdot \nabla g_p(x_i^k)}{\lambda + \mu}$$

previous iteration step

smoothness term

Description of the iteration



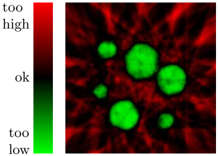
discretizing function

$$y_i^{k+1} \leftarrow x_i^k - \frac{v_i + \alpha \cdot w_i + \mu \cdot G_{0,\sigma}(v_i) \cdot \nabla g_p(x_i^k)}{\lambda + \mu}$$

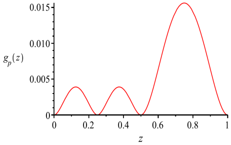
previous iteration step

smoothness term

Description of the iteration



backprojected error of projections



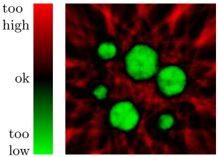
discretizing function

$$y_i^{k+1} \leftarrow x_i^k - \frac{v_i + \alpha \cdot w_i + \mu \cdot G_{0,\sigma}(v_i) \cdot \nabla g_p(x_i^k)}{\lambda + \mu}$$

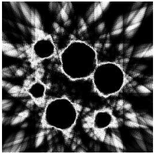
previous iteration step

smoothness term

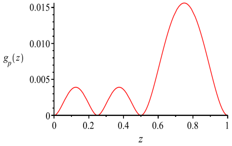
Description of the iteration



backprojected error of projections



discretization weights



discretizing function

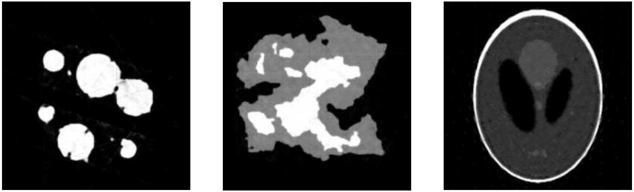
$$y_i^{k+1} \leftarrow x_i^k - \frac{v_i + \alpha \cdot w_i + \mu \cdot G_{0,\sigma}(v_i) \cdot \nabla g_p(x_i^k)}{\lambda + \mu}$$

previous iteration step

smoothness term

Results of the optimization

The result of the optimization process is a semi-continuous reconstruction on which pixel values are somewhat steered towards discrete solutions.



+ Animation

Example of the process

Evaluation of the algorithm

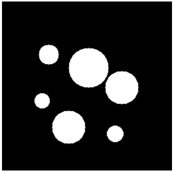
We evaluated the algorithm by running software tests.

- We have chosen two other reconstruction algorithms for comparision.
  - Discrete Algebraic Reconstrction Algorithm (DART)

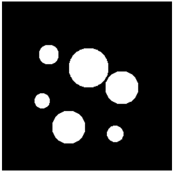
    K.J. Batenburg, J. Sijbers, *DART: a practical reconstruction algorithm for discrete tomography*, IEEE Transactions on Image Processing 20(9), pp. 2542–2553 (2011).
  - A D.C. programming based algorithm, that is capable of reconstructing binary images by minimizing an energy function. (DC)

    T. Schüle, C. Schnörr, S. Weber, J. Hornegger, *Discrete tomography by convex-concave regularization and D.C. programming*, Discrete Applied Mathematics 151, pp. 229–243 (2005).
- We reconstructed a set of software phantoms from their projections with the three given algorithms.
- After the reconstructions we compared the results for evaluation.

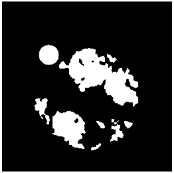
Results



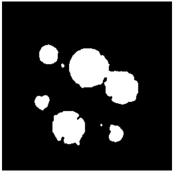
Original image



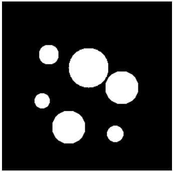
DC, 5 projs.



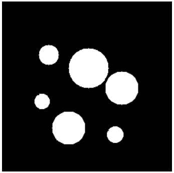
DART, 5 projs.



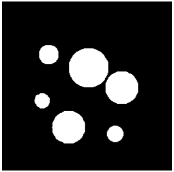
Prop. method, 5 projs.



DC, 6 projs.



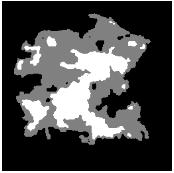
DART, 6 projs.



Prop. method, 6 projs.

Results

Original image



DART, 6 projs.

Prop. method, 6 projs.

DC, 9 projs.

Prop. method, 9 projs.

Results

Original image

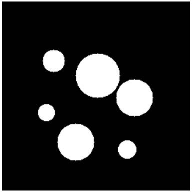
DART, 15 projs.

Prop. method, 15 projs.

DC, 18 projs.

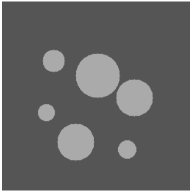
Prop. method, 18 projs.

Numerical results



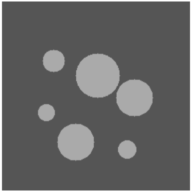
	DC		DART		Prop. method	
projections	Error	Time	Error	Time	Error	Time
2	90.7%	12.1 s.	**85.6 %**	6.6 s.	107.4%	10.1 s.
3	**22.0%**	12.4 s.	52.9%	5.4 s.	30.8%	11.2 s.
4	**1.2%**	13.6 s.	44.9%	8.0 s.	22.4%	11.8 s.
5	**0.3%**	12.5 s.	29.9%	9.5 s.	7.9%	12.7 s.
6	**0.2%**	8.1 s.	**0.2%**	2.7 s.	0.8%	7.6 s.
9	0.2%	6.5 s.	**0.0%**	0.8 s.	0.3%	4.6 s.
12	0.0%	7.2 s.	**0.0%**	0.9 s.	0.1%	4.8 s.
15	0.0%	8.7 s.	**0.0%**	1.2 s.	0.1%	5.8 s.
18	0.0%	8.7 s.	**0.0%**	0.9 s.	0.1%	5.8 s.

Numerical results



	DC		DART		Prop. method	
projections	Error	Time	Error	Time	Error	Time
2	-	-	62.9%	6.7 s.	**52.7%**	10.4 s.
3	-	-	45.1%	8.0 s.	**41.9%**	11.4 s.
4	-	-	43.4%	8.6 s.	**35.4%**	12.2 s.
5	-	-	36.4%	9.4 s.	**26.4%**	13.2 s.
6	-	-	27.0%	10.2 s.	**11.6%**	13.8 s.
9	-	-	**0.7%**	4.5 s.	1.9%	15.6 s.
12	-	-	**0.4%**	14.9 s.	1.0%	11.6 s.
15	-	-	**0.3%**	2.3 s.	0.8%	11.6 s.
18	-	-	**0.1%**	21.3 s.	0.6%	10.9 s.

Numerical results



	DC		DART		Prop. method	
projections	Error	Time	Error	Time	Error	Time
2	-	-	**84.4%**	6.7 s.	85.7%	9.3 s.
3	-	-	**77.3%**	8.2 s.	82.5%	6.0 s.
4	-	-	**75.3%**	8.8 s.	81.0%	8.0 s.
5	-	-	**73.3%**	9.7 s.	74.2%	10.2 s.
6	-	-	74.1%	10.2 s.	**70.0%**	12.7 s.
9	-	-	57.0%	12.6 s.	**46.8%**	14.7 s.
12	-	-	33.9%	14.5 s.	**24.8%**	11.4 s.
15	-	-	22.0%	18.0 s.	**16.3%**	8.6 s.
18	-	-	15.7%	20.8 s.	**14.0%**	8.0 s.

Summary

- Based on the results, the proposed method can compete with the other two algorithms in both speed and accuracy of the results.
- With reconstructions of images containing at least 3 intensity levels from few projections, it could outperform the other two methods.
- There are several possible ways for improvement and alternative applications of the algorithm.

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