Quantum theory of light-matter interaction: Fundamentals

Lecture 2
Linear dipole oscillator, Lorentz model

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BEFEKTETÉS A JÖVŐBE

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The polarization **P** of matter plays a central role in classical electrodynamics. While it is usually generated by an external electric field **E** in a medium with polarizable atoms, it is also a source of a secondary electric field. For low field strengths, and for isotropic media **P** and **E** oscillating at the same frequency ω are proportional $\mathbf{P}(\omega) = \epsilon_0\chi(\omega)\mathbf{E}(\omega)$. As we have seen in Lecture 1. the dielectric susceptibility $\chi = \chi' + i\chi''$ is a complex quantity, and therefore the proportionality has a twofold effect. (1) It changes the phase velocity of the electromagnetic wave in the medium, and (2) as $\chi'' \neq 0$, **P** does not oscillate in the same phase as **E**, resulting in the absorbtion of the wave in the medium among ordinary circumstances.

The actual functional form of $\chi(\omega)$ depends on the properties of the individual responses of the microscopic constituents to the excitation $\mathbf{E}(\omega)$, as well as on the structure how these atoms build up the macroscopic medium. Here we consider a simple, non-quantum mechanical model of the microcopic response of an atom to the external field, that can provide a first impresssion about the underlying physical mechanisms.

The Lorentz model (established before the advent of quantum mechanics) assumes that there is a single active electron in the atom considered to be a classical point charge $e_0(< 0)$ of mass m, bound to the nucleus by a harmonic force $F = -Dx := -m\omega_0^2 x$,
x labels the deviation from the equilibrium position with the center of charge at the nucleus.

The accelerated charge radiates and loses energy. Usually this is taken into account as a damping force $-\gamma\dot{x}(t)$
For small x this linear model is sufficient, the oscillator is subject to an external electric field, supposing that the latter is not too strong.

The equation of motion for this particle under the action of a linearly polarized time varying electric field $E(t)$:

$$\ddot{x}(t) + \gamma \dot{x}(t) + \omega_0^2 x(t) = \frac{e_0}{m} E(t) \tag{Osc}$$

The second term contains the damping of the oscillator.
Quantum mechanically this decay is due to spontaneous emission and collisions to be discussed in later lectures.
γ is usually at least three orders of magnitude smaller than ω_0. For optical fields in the visible range $\omega_0 \sim 10^{15}$ 1/s, while $\gamma \sim 10^{12}$ 1/s for atoms in a dense medium and only of the order of 10^8 1/s for an isolated atom.

Lecture 3. will generalize the classical linear model (Osc) by adding nonlinear forces proportional to x^2 and x^3
These generate coupling between field modes producing important effects such as sum and difference frequency generation and phase conjugation. (Osc) and its nonlinear extensions allow us to see the "atom"-field interactions in a simple classical context before we consider them in their more realistic, but complex, quantum form.

We present the solution of (Osc) for a driving field

$$E(t) = E_0 \cos \omega t = \frac{1}{2} E_0 e^{-i\omega t} + c.c.$$

where E_0 is a constant real amplitude.

The solution of

$$\ddot{x}(t) + \gamma \dot{x}(t) + \omega_0^2 x(t) = \frac{1}{2} E_0 e^{-i\omega t} \qquad \text{(Osc)}$$

is a sum of two terms. One is the general solution of the corresponding homogeneous equation and has the form $A \cos(\omega_1 t + \delta) e^{-\gamma t/2}$, where $\omega_1 \approx \omega_0$. This is a transient and decays very fast with the decay constant γ. Unless we have very short pulses shorter than $1/\gamma$ this term does not play any role.

The other term, the particular solution of the inhomogeneous equation (Osc), is the stationary solution which oscillates with the frequency of the exciting field:

$$x(t) = \frac{e_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \frac{1}{2} E_0 e^{-i\omega t}$$

The dipole moment corresponding to this solution is

$$D_a = e_0 x(t) = \underbrace{\frac{e_0^2}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}}_{\beta(\omega):\text{ polarizability}} \frac{1}{2} E_0 e^{-i\omega t}$$

The dipole moment density is :$P_0 = \mathcal{N} D_a$, and the dielectric susceptibility χ defined through

$$P_0 = \chi \epsilon_0 E_0$$

is given now by

$$\chi(\omega) = \mathcal{N}\frac{e_0^2}{\epsilon_0 m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

This is connected to the relative dielectric constant $\epsilon(\omega) = 1 + \chi(\omega)$, and to the complex index of refraction, (in a nonmagnetic material where $\tilde{n} = \sqrt{\epsilon}$) by

$$\tilde{n}(\omega) = \sqrt{1 + \chi(\omega)} \approx 1 + \chi(\omega)/2 = 1 + \mathcal{N}\frac{e_0^2}{2\epsilon_0 m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

The $\sqrt{1+\chi(\omega)} \approx 1+\chi(\omega)/2$ approximation is valid in atomic vapors where $\mathcal{N}$ is low enough.

$$\tilde{n}(\omega) = n + i\kappa$$
$$n(\omega) = 1 + \mathcal{N}\frac{e_0^2}{2\epsilon_0 m}\frac{(\omega_0^2-\omega^2)^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$
$$\kappa(\omega) = \mathcal{N}\frac{e_0^2}{2\epsilon_0 m}\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

This gives for the solution of a monochromatic plane wave

$$E^+(z,t) = E_0^+ e^{-i\omega t} e^{in\frac{\omega}{c}z} e^{-\alpha(\omega)z/2}$$

i.e. an index of refraction $n(\omega)$ and amplitude absorbtion coefficient $\alpha(\omega)/2 = \kappa\frac{\omega}{c}$, giving the Bouguer-Lambert-Beer law for the intensity again:

$$I(z) = I(0)e^{-\alpha(\omega)z}$$

The usual procedure for solving the Lorentz model – as shown above – is applicable if E_0 is constant.
If $E_0(t)$ is time dependent, then we may apply another method.
It will be useful especially close to resonance, and it is also applicable in the quantum version of the problem, where the harmonic oscillator model fails, the atom can be brought to an excited stationary state.
Suppose that the electric field at the location of the atom is

$$E^+ = \frac{1}{2} E_0(t) e^{-i\omega t}$$

where $E_0(t)$ is slowly varying in time compared to $e^{-i\omega t}$. Accordingly we assume the solution of (Osc) in the form

$$x(t) = \frac{1}{2} x_0 X(t) e^{-i\omega t}$$

where x_0 is a constant, X is dimensionless and $|\dot{X}| \ll \omega |X|$ which is equivalent to the slowly varying envelope approximation SVEA for the amplitude.

Substitute $x(t)$ into (Osc) and according to SVEA omit the terms $\ddot{X}$, and $\gamma\dot{X}$ compared to the others.
Also with $\omega \approx \omega_0, \quad \omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega\Delta$, where $\Delta = \omega_0 - \omega$,
the resulting equation

$$\dot{X} + i\Delta X + \frac{\gamma}{2}X = \frac{i}{2\omega x_0}E_0(t)$$

Writing $X = U - iV$ we get:

$$\dot{U} = -\Delta V - \frac{\gamma}{2}U$$
$$\dot{V} = \Delta U - \frac{\gamma}{2}V - \frac{e_0}{2mx_0\omega}E_0(t)$$

Quantum mechanics (QM)

We shall obtain later a QM *generalization* of the above system, and it will be completed by a third equation describing the possibility of transferring the excitation between different quantized energy levels. In that system, called the *optical Bloch equations,* the notation $T_2 = 2/\gamma$ will be used. Borrowing that convention from the quantum treatment, our "classical optical Bloch equations" take the form:

$$\dot{U} = -\Delta V - \frac{U}{T_2} \tag{U}$$

$$\dot{V} = \Delta U - \frac{V}{T_2} - \frac{e_0}{2mx_0\omega}E_0(t). \tag{V}$$

Remarks

- The QM version shall include the possibility to invert the atom not present in the Lorentz model.
- We have seen in the previous lecture, that the field amplitude was driven by the imaginary part of the polarization (i.e., V).
- The reverse is also seen now: the field $E_0(t)$ drives the imaginary (in-quadrature) part V of the slowly varying polarization amplitude.
- The in-phase part, U plays role only if there is no exact resonance.

Questions

1. What are the basic assumptions of the Lorentz model?
2. What physical mechanisms are modeled by the damping term?
3. For optical excitations, what is the order of magnitude of ω_0/γ?
4. What is the form of the general solution of Eq. (Osc)?
5. What is the ω dependence of the index of refraction resulting from the Lorentz model?
6. Can the Bouguer-Lambert-Beer law be derived using the Lorentz model?

Reference

1. P. Meystre and M. Sargent, *Elements of Quantum Optics*, Spinger (Berlin, Heidelberg) (2007).