Mathematics

Lecture and practice

Norbert Bogya

University of Szeged, Bolyai Institute

2017

Course details

- I am: Norbert Bogya
  - Email: nbogya@math.u-szeged.hu
  - Office address: Aradi vértanúk tere 1., H-6720 Szeged, Hungary, Bolyai Institute, Room M05.
- The lecture and the practice will not be held separately, mainly we are going to solve exercises, but you will receive grades for both.
- To attend the classes is not obligatory, but ...

Grading

Starting from the second lesson, at the beginning of the lessons there will be a short, 10 min **Quiz** (8-10 altogether) worth 5 points (each), however only the best 6-8 quizzes will count towards your **practice grade**. Grading will be done according to the following table.

80-100%	5
64-79%	4
47-63%	3
30-46%	2
0-29%	1

The **lecture grade** will be based on the 2 **midterms** (60 min long, 25 points each) which will be held on the 6-7th and 12-13th week (no quizzes at these times). The grading table is just like before.

Plan (Probably, it will change.)

Week 1. Set theory
Week 2. Quiz 1; Functions
Week 3. Quiz 2; Limit
Week 4. Quiz 3; Limit
Week 5. Quiz 4; Differentiation
Week 6. Quiz 5; Differentiation
Week 7. Midterm 1; Differentiation
Week 8. Quiz 6; Integration
Week 9. Quiz 7; Sequences
Week 10. Quiz 8; Series
Week 11. Quiz 9; Linear algebra
Week 12. Quiz 10; Linear programming
Week 13. Midterm 2

References

- H. Anton and C. Rorres, *Elementary Linear Algebra with Applications*, 10th ed.
- B. Kolman and R. E. Beck, *Elementary Linear Programming with Applications*, 2nd ed.
- K. Leung and D. L. chue Chen, *Elementary Set Theory*, Hong Kong University Press.
- B. Shillito, *Introduction to higher mathematics*.
- G. B. Thomas, M. D. Weir, J. Hass, and F. R. Giordano, *Thomas' Calculus*, 11th ed.

Basic set theory

Basic "definitions"

A set is a **well-defined** collection of distinct objects. The objects that make up a set (also known as the **elements** or members of a set) can be anything.

Notation: $A = \{1, 2, 3, 4, 5\}$.

Membership: there is fundamental "belonging to" connection between objects and sets: $5 \in A$, $7 \notin A$

We can say: x is in H, x is an element of H, H contains x ...

Definition

Sets A and B are **equal** if and only if they have precisely the same elements.

$\{2, 3, 5, 7\}$ = set of 1-digit positive prime numbers

Basic "definitions"

A set can be described by

(1) a rule or semantic **description**:
A is the set whose members are the first four positive integers
B is the set of colours of the French flag

(2) **listing** each member:
$A = \{2, 3, 1, 4\}$
$B = \{\text{red}, \text{white}, \text{blue}\}$

(3) a mathematical **formula**:
$A = \{x \in \mathbb{Z} \colon 1 \leq n < 5\}$
$F = \{n^2 - 4 \mid n \text{ is even and } 3 \leq n \leq 8\}$

More than sets

Remarks.

- A set is irrelevant to multiplicity.

$$\{a, b, b, c, c, c\} = \{a, b, c\}$$

- A set is irrelevant to ordering.

$$\{1, 4, 3, 2\} = \{1, 2, 3, 4\}$$

If we consider multiplicity, we get a multiset.
If we consider ordering, we get ordered set.

Empty set

Definition

The set which contains no element is called empty set.

Theorem

There is one and only one set, which contains no element.

Notation: $\emptyset$.

$$\begin{aligned} \emptyset &= \text{set of red horses with 7 heads} \\ &= \{x \mid x \text{ is an even prime and } x > 20\} \end{aligned}$$

Combining sets

Operations are used for construct new objects from same type of different ones. For example: addition of numbers.

We can take several operations with sets:

- intersection
- union
- complement
- difference
- symmetric difference

Combining sets

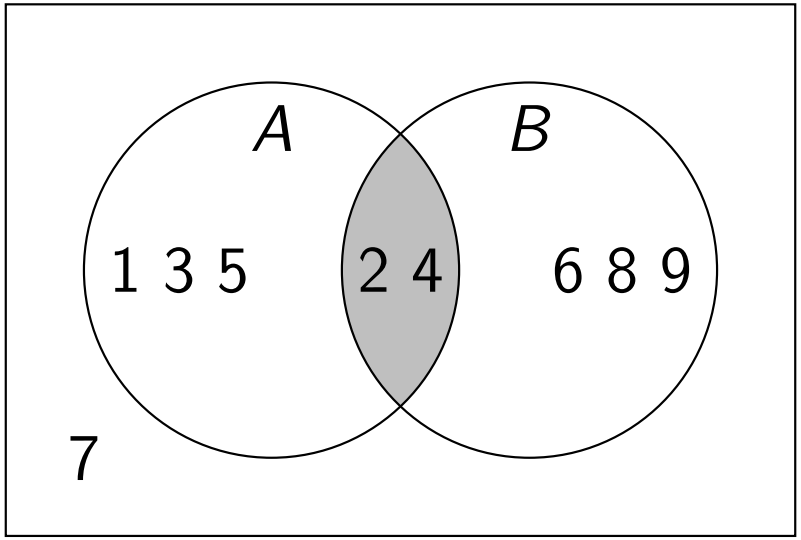
Intersection operator

Definition

$A \cap B = \{x \mid x \in A \textbf{ and } x \in B\}$

Example

$A = \{1, \mathbf{2}, 3, \mathbf{4}, 5\}$, $B = \{\mathbf{2}, \mathbf{4}, 6, 8, 9\}$, $A \cap B = \{\mathbf{2}, \mathbf{4}\}$



Combining sets

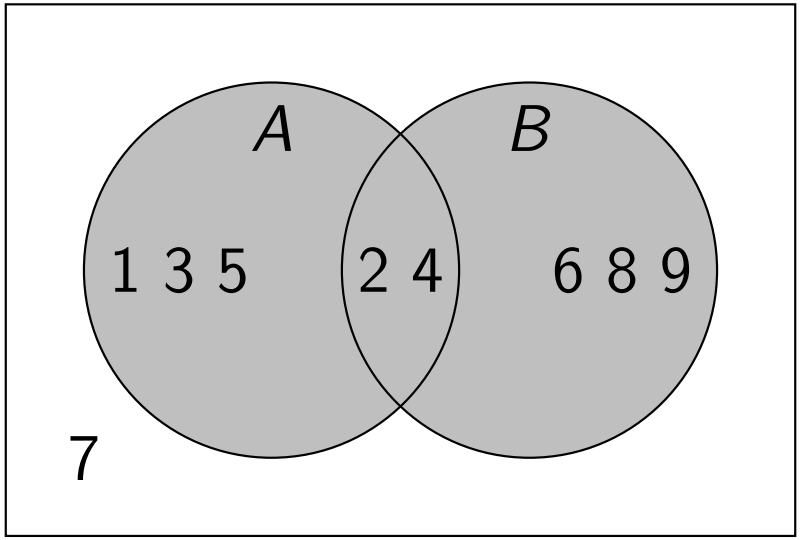
Union operator

Definition

$A \cup B = \{x \mid x \in A \textbf{ or } x \in B\}$

Example

$A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 9\}$, $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 9\}$



Combining sets

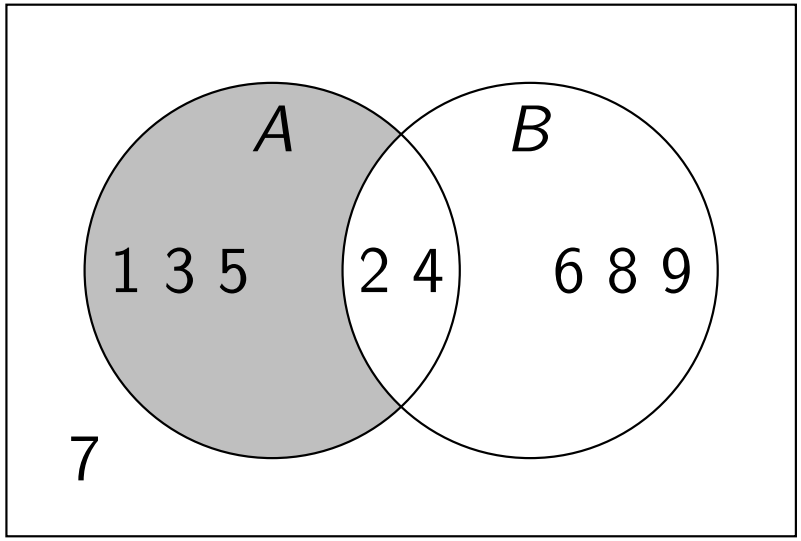
Difference operator

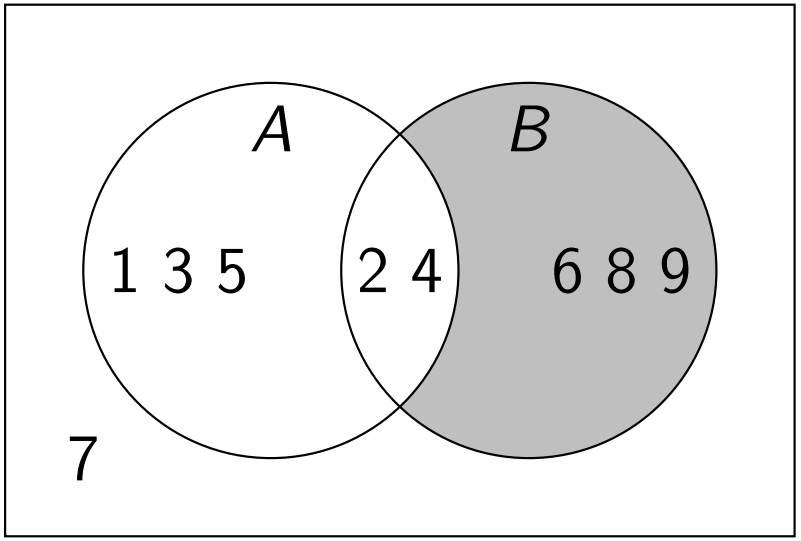
Definition

$A \setminus B = \{x \mid x \in A \textbf{ but } x \notin B\}$

Example

$A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 9\}$,
$A \setminus B = \{1, 3, 5\}$, $B \setminus A = \{6, 8, 9\}$





Combining sets

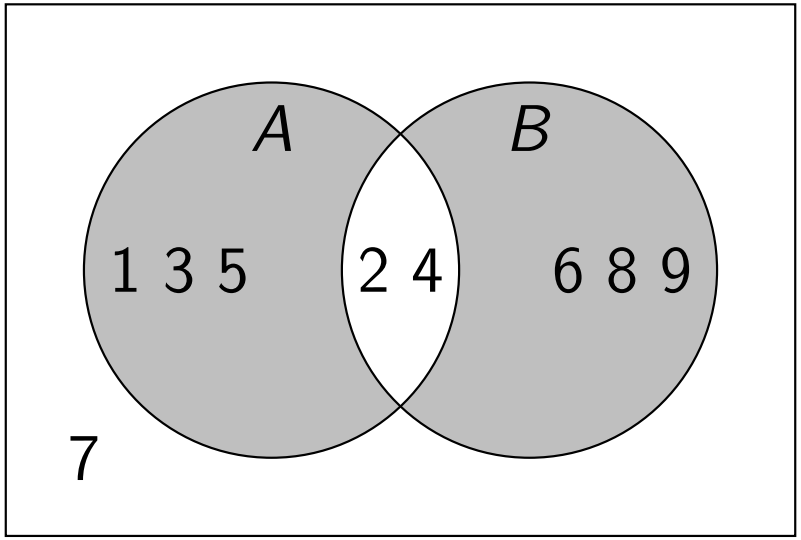
Symmetric difference operator

Definition

$A \triangle B = \{x \mid x \in A \setminus B \textbf{ or } x \in B \setminus A\} = (A \cup B) \setminus (A \cap B)$

Example

$A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 9\}$, $A \triangle B = \{1, 3, 5, 6, 8, 9\}$



Combining sets

Complement operator

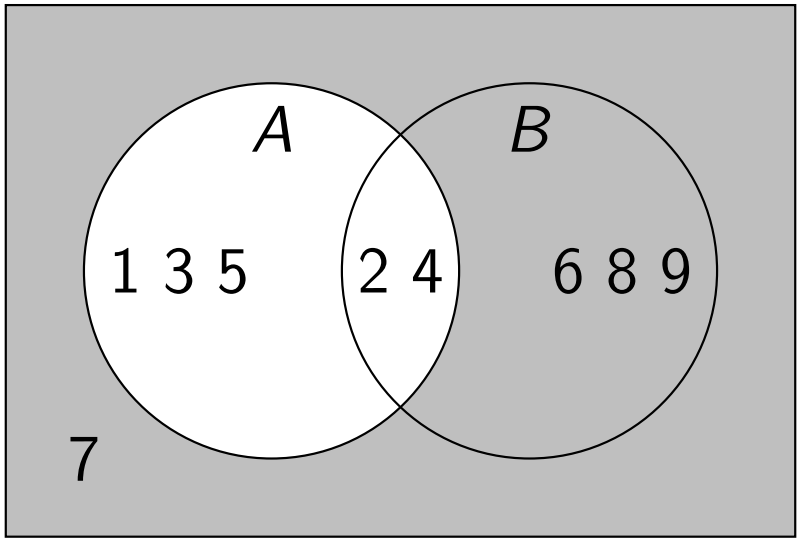
Universal set: everything we care about in the context of our problem.

Definition

$A^c = \overline{A} = \{x \mid x \in U \text{ but } x \notin A\}$

Example

$A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 9\}$, $\overline{A} = \{6, 7, 8, 9\}$



Practice

Exercise

Let $U = \{a, b, c, d, e\}$ be the universal set and $A = \{a, b, c, d\}$, $B = \{d, e\}$, $C = \{a, b, e\}$. Give the elements of the following sets.

$$A \cup B, \quad A \cap B, \quad \overline{B}, \quad A \setminus B, \quad B \setminus A,$$
$$A \triangle B, \quad (A \triangle \overline{C}) \setminus \overline{B}, \quad (C \setminus A) \triangle B$$

Exercise

Illustrate the following sets.

$$[0, 2] \cap (1, 3), \quad [0, 3) \cup (-1, 2], \quad (0, 5] \setminus [1, 2]$$

Properties of operations

Theorem

For all sets A, B, C we have the following statements:

$$A \cap A = A, \qquad A \cup A = A,$$
$$A \cap B = B \cap A, \qquad A \cup B = B \cup A,$$
$$(A \cap B) \cap C = A \cap (B \cap C), \qquad (A \cup B) \cup C = A \cup (B \cup C),$$
$$(A \cup B) \cap A = A, \qquad (A \cap B) \cup A = A,$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \qquad (A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Properties of operations

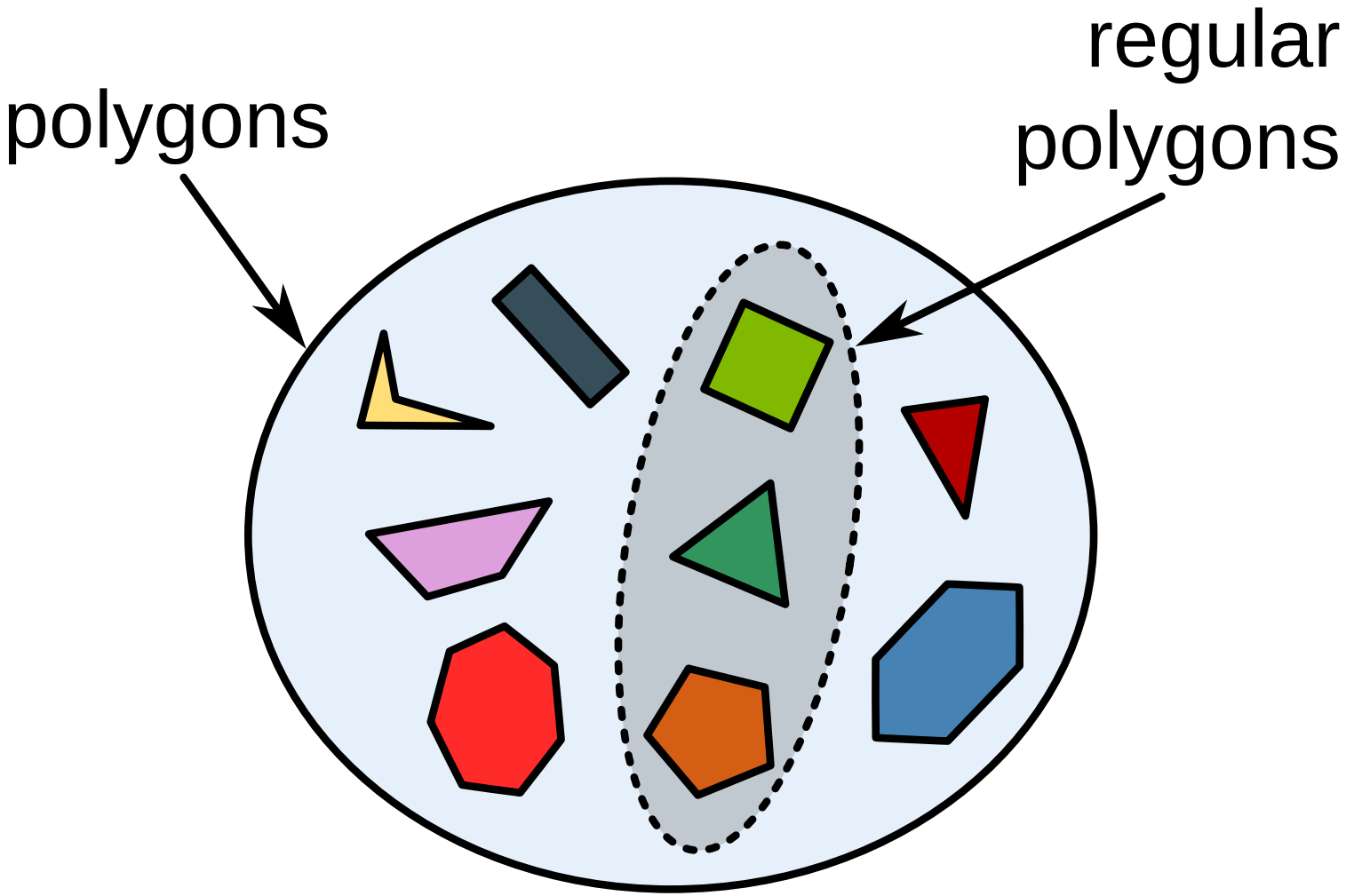
Theorem

For all sets $A, B\,(\subseteq U)$ we have the following statements:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}, \qquad \overline{A \cup B} = \overline{A} \cap \overline{B},$$
$$\overline{\overline{A}} = A,$$
$$A \cap \overline{A} = \emptyset, \qquad A \cup \overline{A} = U,$$
$$A \cap U = A, \qquad A \cup U = U,$$
$$A \cap \emptyset = \emptyset, \qquad A \cup \emptyset = A.$$

The equalities in the first row are called **De Morgan's laws**.

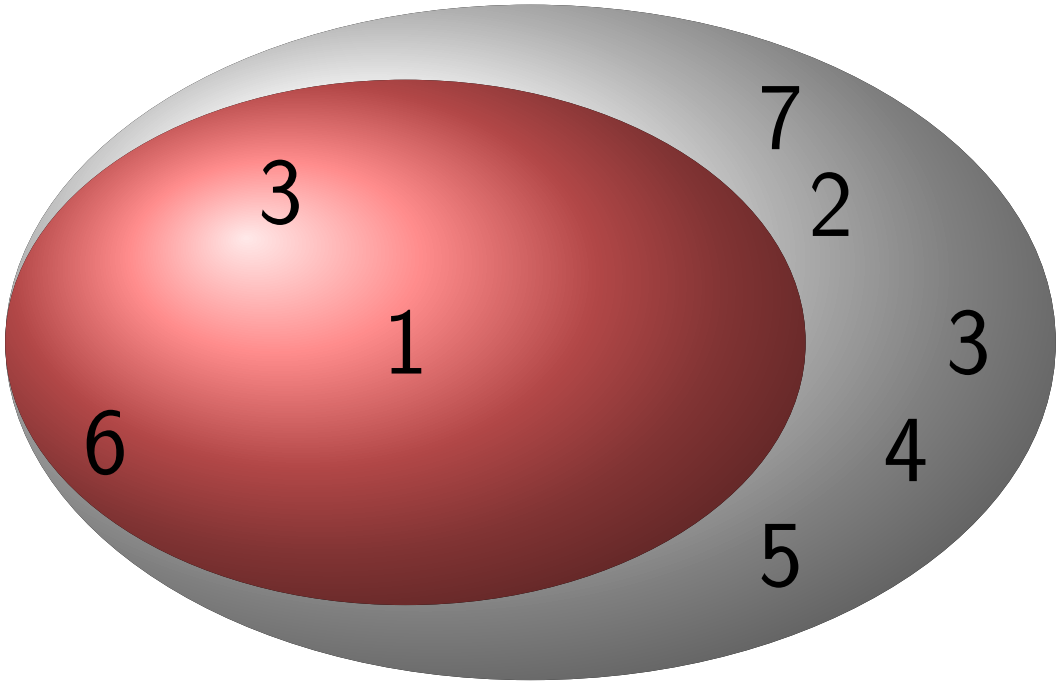
Subsets



Subsets

Definition

If every member of set A is also a member of set B, then A is said to be a **subset** of B, written $A \subseteq B$.



$$\{1, 3, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$$

Subsets

Theorem

Every H set has trivial subsets:

- $\emptyset \subseteq H$
- $H \subseteq H$

Definition

Let A be a set. The set of all subsets of A is called **power set** of A and it is denoted by $\mathcal{P}(A)$. So

$$\mathcal{P}(A) = \{X : X \subseteq A\}.$$

Questions

$\mathcal{P}(\{1,2\})=?$ $\quad$ $\mathcal{P}(\{a,1,5\})=?$ $\quad$ $\mathcal{P}(\emptyset)=?$ $\quad$ $\mathcal{P}(\mathcal{P}(\emptyset)) =?$

Cardinality

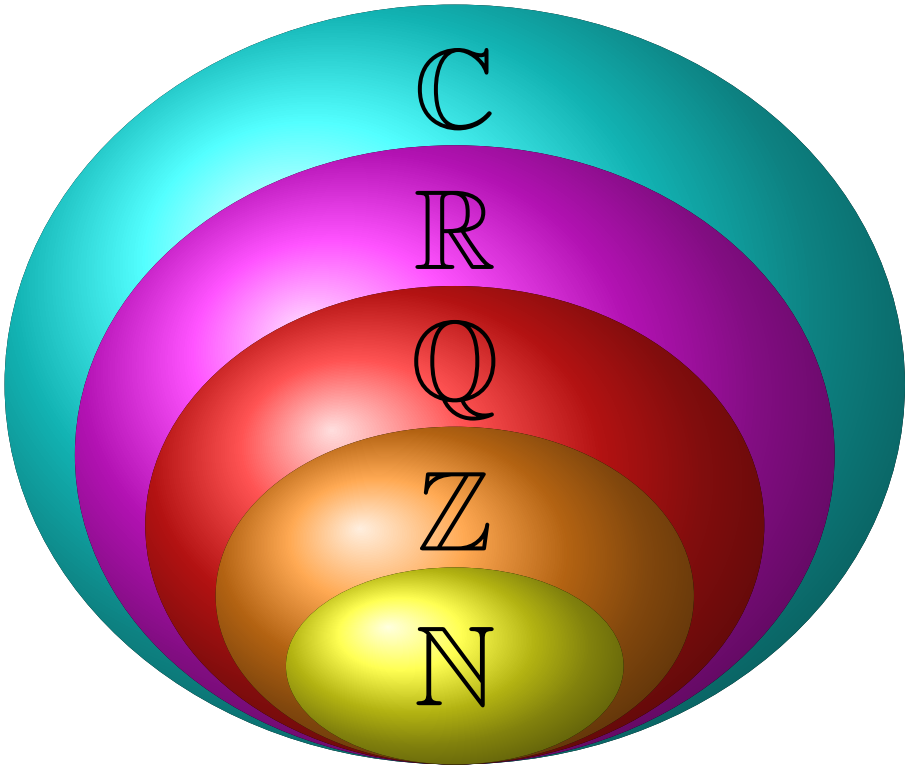
A set has finite or infinitely many element. This mean a cardinality of a set can be **finite** or **infinite**. The cardinality of the set H is denoted by $|H|$.

Theorem

If H is a finite set and $|H| = n$, then $|\mathcal{P}(H)| = 2^n$.

This theorem holds for all cardinality, but to prove that we need deeper knowledge in mathematics.

Number sets



Natural numbers:
$\mathbb{N} = \{1, 2, 3, \ldots\}$

Integers:
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Rational numbers:
$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}\right\}$

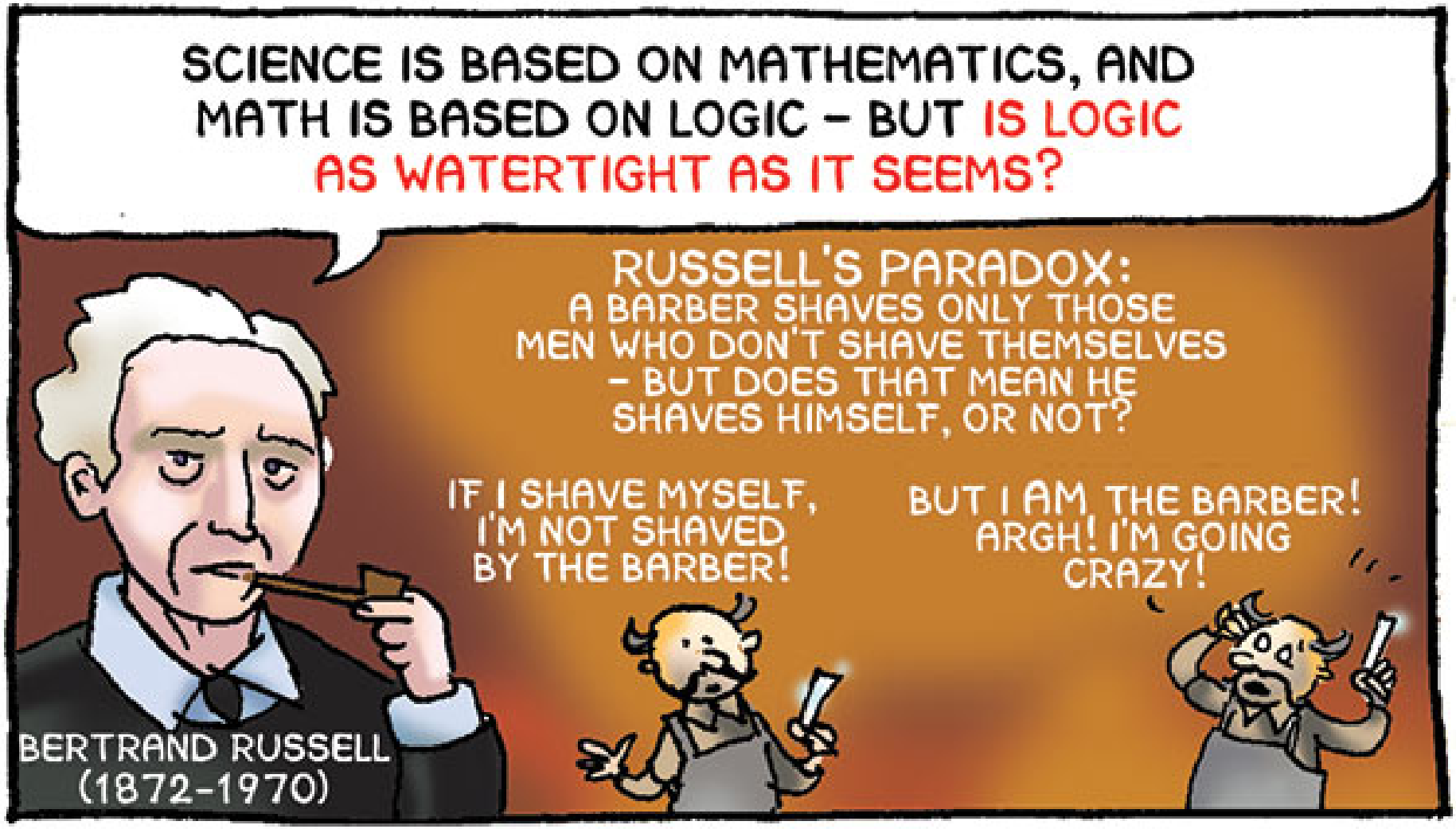
Real numbers: $\mathbb{R}$

Complex numbers: $\mathbb{C}$

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

These number sets are infinite sets, but $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}| = |\mathbb{C}|$.

Barber paradox



Barber paradox

Let R be the set of all sets, that DO NOT contain themselves.

$$R = \{A \mid A \notin A\}$$

Paradox:

- $R \in R \Longrightarrow R \notin R$
- $R \notin R \Longrightarrow R \in R$

This paradox can be resolved by axiomatic set theory, but it needs deeper mathematical knowledge.

Barber paradox