Mathematics
Lecture and practice

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Limit

example
approach analytic
defined
infinity
substitution
finite
left-hand
one-sided
continuity
factor
infinite
lim
zeros
increasing
domain
right-hand
exist
canceling
interior
large
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continuous
The word *limes* was used by Latin writers to denote a marked or fortified frontier. This term has been adapted and used by modern historians as an equivalent for the frontiers of the Roman Empire.
Motivating example

How does the function

\[ f(x) = \frac{x^2 - 1}{x - 1} \]

behave near \( x = 1 \)?

**Step 1.** Investigation of the domain: the domain of the function is the real numbers except \( x = 1 \).
Step 2. We substitute some values below and above 1.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>0.5</th>
<th>0.9</th>
<th>0.99</th>
<th>0.999</th>
<th>0.999999</th>
<th>...</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1</td>
<td>1.5</td>
<td>1.9</td>
<td>1.99</td>
<td>1.999</td>
<td>1.999999</td>
<td>...</td>
<td>X</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1</th>
<th>...</th>
<th>1.000001</th>
<th>1.0001</th>
<th>1.001</th>
<th>1.01</th>
<th>1.1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>...</td>
<td>2.000001</td>
<td>2.0001</td>
<td>2.001</td>
<td>2.01</td>
<td>2.1</td>
<td>2.5</td>
<td>3</td>
</tr>
</tbody>
</table>

Step 3. We can try to graph the function with computer.
Step 4. Try to do some mathematics...

\[ f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1, \quad (x \neq 1) \]

We say that \( f(x) \) approaches the limit 2 as \( x \) approaches 1, and write

\[ \lim_{x \to 1} f(x) = 2 \quad \text{or} \quad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2. \]
Remark. The limit value does not depend on how the function is defined at $x_0$. 

$$f(x) = x + 1$$

$$g(x) = \frac{x^2 - 1}{x - 1}$$

$$h(x) = \begin{cases} 
\frac{x^2 - 1}{x - 1}, & x \neq 1 \\
1, & x = 1 
\end{cases}$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2$$
What about these?

\[
f(x) = \begin{cases} 
0, & x < 0 \\
1, & x \geq 0 
\end{cases}
\]

\[
g(x) = \begin{cases} 
\frac{1}{x}, & x \neq 0 \\
0, & x = 0 
\end{cases}
\]

(1) Both of the two functions defined over real numbers (no exclusion).

(2) None of the two functions has limit at \( x_0 = 0 \).
**Limit**

**Definition**

Let $f(x)$ be defined on an open interval about $x_0$, except at $x_0$ itself. We say that the limit of $f(x)$ as $x$ approaches $x_0$ is the number $L$, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $x$,

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$$
To have a limit $L$ as $x$ approaches $c$, a function must be defined on both sides of $c$ and its values $f(x)$ must approaches $c$ from either side. Because of this, ordinary limits are called two-sided.

If $f$ fails to have a two-sided limit $c$, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a right-hand limit. From the left, it is a left-hand limit.
One-sided limit

\[ y = f(x) \]

\[
\begin{align*}
\lim_{x \to 2^+} f(x) &= 2 \\
\lim_{x \to 2^-} f(x) &= 4
\end{align*}
\]

\[ f(2) = 3 \]
One-sided limit

**Definition**

If $f(x)$ is defined on an interval $(x_0, c)$, where $x_0 < c$ and approaches arbitrarily close to $L$ as $x$ approaches $x_0$ from within that interval, then $f$ has **right-hand limit** $L$ at $x_0$.

$$\lim_{x \to x_0^+} f(x) = L$$

**Definition**

If $f(x)$ is defined on an interval $(a, x_0)$, where $a < x_0$ and approaches arbitrarily close to $M$ as $x$ approaches $x_0$ from within that interval, then $f$ has **left-hand limit** $M$ at $x_0$.

$$\lim_{x \to x_0^-} f(x) = M$$
A function \( f(x) \) has a limit as \( x \) approaches \( c \) if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

\[
\lim_{{x \to c}} f(x) = L \iff \lim_{{x \to c^+}} f(x) = L \text{ and } \lim_{{x \to c^-}} f(x) = L.
\]
One-sided limit

\[ y = f(x) \]

\[ \lim_{x \to 2^+} f(x) = 2 \]

\[ \lim_{x \to 2^-} f(x) = 4 \]

\[ \lim_{x \to 2} f(x) \text{ doesn't exist} \]
What about these?

\[ f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \]

\[ g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \]

(1) \( f(0) = 1, \lim_{x \to 0^-} f(x) = 0, \lim_{x \to 0^+} f(x) = 1 \)

(2) \( g(0) = 0, \lim_{x \to 0} g(x) = ? \)
Find the one- and two-sided limits at $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$. 
Finding limit algebraically
By substitution

Exercise

Find the following limits.

(a) \( \lim_{{x \to 2}} 4 = \)

(b) \( \lim_{{x \to -13}} 4 = \)

(c) \( \lim_{{x \to 3}} x = \)

(d) \( \lim_{{x \to 2}} (5 - 2x) = \)

(e) \( \lim_{{x \to -1}} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \)

(f) \( \lim_{{x \to -2}} \frac{3x + 4}{x + 5} = \)
Finding limit algebraically
(Creating and) canceling a common factor

Exercise

Find the following limits.

(a) \( \lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \)

(b) \( \lim_{x \to 0} \frac{x^2(x^2 - 3x + 2)}{x^2 + x} = \)

(c) \( \lim_{x \to -2} \frac{-2x - 4}{x^3 + 2x^2} = \)
What about $f(x) = \frac{1}{x}$?

- As $x \rightarrow 0^+$ the values of $f$ grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number $B$, however large, the values of $f$ become larger still.

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$
What about \( f(x) = \frac{1}{x} \)?

- As \( x \to 0^- \) the values of \( f \) become arbitrarily large and negative. Given any negative real number \(-B\), the values of \( f \) eventually lie below \(-B\).

\[
\lim_{x \to 0^-} \frac{1}{x} = -\infty
\]
Exercises

(1) \[ \lim_{x \to 2} \frac{(x - 2)^2}{x^2 - 4} = \]

(2) \[ \lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \]

(3) \[ \lim_{x \to 2^+} \frac{x - 3}{x^2 - 4} = \]

(4) \[ \lim_{x \to 2^-} \frac{x - 3}{x^2 - 4} = \]

(5) \[ \lim_{x \to 2} \frac{x - 3}{x^2 - 4} = \]

(6) \[ \lim_{x \to 2} \frac{2 - x}{(x - 2)^3} = \]

Conclusion

Rational functions can behave in various ways near zeros of their denominators.
What about \( f(x) = \frac{1}{x} \)?

- When \( x \) is positive and becomes increasingly large, \( 1/x \) becomes increasingly small.
  \[
  \lim_{x \to \infty} \frac{1}{x} = 0
  \]

- When \( x \) is negative and its magnitude becomes increasingly large, \( 1/x \) again becomes small.
  \[
  \lim_{x \to -\infty} \frac{1}{x} = 0
  \]
Finite limits as $x \to \pm \infty$

The symbol for infinity ($\infty$) does not represent a real number. We use $\infty$ to describe the behaviour of a function when the values in its domain or range outgrow all finite bounds.

**Definition**

We say that $f(x)$ has the limit $L$ as $x$ approaches infinity (minus infinity) and write

$$
\lim_{x \to \infty} f(x) = L, \quad \lim_{x \to -\infty} f(x) = L
$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $M$ ($N$) such that for all $x$,

$$
x > M, \quad (x < N) \quad \implies \quad |f(x) - L| < \varepsilon.
$$
Exercise

Find the limits of $f(x) = \frac{1}{x-1}$ at $x_0 = 1$ and at infinities.

**Geometric solution.**

\[
\lim_{x \to 1^+} f(x) = \infty
\]

\[
\lim_{x \to 1^-} f(x) = -\infty
\]

\[
\lim_{x \to \infty} f(x) = 0
\]

\[
\lim_{x \to -\infty} f(x) = 0
\]
Exercise

Find the limits of \( f(x) = \frac{1}{x-1} \) at \( x_0 = 1 \) and at infinities.

Analytic solution.

**Theorem**

\[
\lim_{x \to \pm \infty} \frac{1}{x} = 0, \quad \lim_{x \to 0^+} \frac{1}{x} = \infty, \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty
\]

- \( x \to \infty \quad \Rightarrow \quad x - 1 \to \infty \quad \Rightarrow \quad \frac{1}{x - 1} \to 0 \)
- \( x \to -\infty \quad \Rightarrow \quad x - 1 \to -\infty \quad \Rightarrow \quad \frac{1}{x - 1} \to 0 \)
- \( x \to 1^+ \quad \Rightarrow \quad x - 1 \to 0^+ \quad \Rightarrow \quad \frac{1}{x - 1} \to \infty \)
- \( x \to 1^- \quad \Rightarrow \quad x - 1 \to 0^- \quad \Rightarrow \quad \frac{1}{x - 1} \to -\infty \)
Exercises

(1) \( \lim_{x \to \infty} \left( 5 + \frac{1}{x} \right) = \)

(2) \( \lim_{x \to -\infty} \frac{\pi \sqrt{3}}{x^2} = \)

(3) \( \lim_{x \to \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \)

(4) \( \lim_{x \to -\infty} \frac{11x + 2}{2x^3 - 1} = \)

(5) \( \lim_{x \to \infty} 7 - \frac{8}{x^2} = \)

(6) \( \lim_{x \to \infty} \frac{7x^3}{x^3 - 3x^2 + 6x} = \)

(7) \( \lim_{x \to \infty} \frac{2x^3}{5x^2 + 6x} = \)

(8) \( \lim_{x \to \infty} \frac{2 - x^5}{x^3 + 3x} = \)
Exercises

(1) \[ \lim_{y \to 2} \frac{y + 2}{y^2 + 5y + 6} \]

(2) \[ \lim_{x \to 4} \frac{4x - x^2}{2 - \sqrt{x}} \]

(3) \[ \lim_{x \to -2^+} \frac{|x + 3|}{x + 2} \]

(4) \[ \lim_{x \to \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}} \]

(5) \[ \lim_{x \to \infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4} \]
Continuity

\[ y = h(x) \]
Continuity

**Definition**
A function \( y = f(x) \) is continuous at an interior point \( c \) of its domain if
\[
\lim_{x \to c} f(x) = f(c).
\]

**Definition**
A function \( y = f(x) \) is continuous at a left endpoint \( a \) or at a right endpoint \( b \) of its domain if
\[
\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}
\]

**Definition**
A function is continuous if it is continuous at every point of its domain.
Exercise

Is the function

\[ f(x) = \begin{cases} 
\frac{x^2 - 4x + 4}{x^2 + x - 6}, & \text{if } x < 2 \\
0, & \text{if } x = 2 \\
\frac{x^2 - 3x + 2}{x^2 - 4x + 4}, & \text{if } x > 2 
\end{cases} \]

continuous at the point \( x = 2 \)?