Mathematics
Lecture and practice

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Functions
Motivating example

Musical notes are pressure waves in the air that can be recorded. The data in Table 1.2 give recorded pressure displacement versus time in seconds of a musical note produced by a tuning fork.

<table>
<thead>
<tr>
<th>Time</th>
<th>Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00091</td>
<td>-0.080</td>
</tr>
<tr>
<td>0.00108</td>
<td>0.200</td>
</tr>
<tr>
<td>0.00125</td>
<td>0.480</td>
</tr>
<tr>
<td>0.00144</td>
<td>0.693</td>
</tr>
<tr>
<td>0.00162</td>
<td>0.816</td>
</tr>
<tr>
<td>0.00180</td>
<td>0.844</td>
</tr>
<tr>
<td>0.00198</td>
<td>0.771</td>
</tr>
<tr>
<td>0.00216</td>
<td>0.603</td>
</tr>
<tr>
<td>0.00234</td>
<td>0.368</td>
</tr>
<tr>
<td>0.00253</td>
<td>0.099</td>
</tr>
<tr>
<td>0.00271</td>
<td>-0.141</td>
</tr>
<tr>
<td>0.00289</td>
<td>-0.309</td>
</tr>
<tr>
<td>0.00307</td>
<td>-0.348</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00325</td>
<td>-0.248</td>
</tr>
<tr>
<td>0.00344</td>
<td>-0.041</td>
</tr>
<tr>
<td>0.00362</td>
<td>0.217</td>
</tr>
<tr>
<td>0.00379</td>
<td>0.480</td>
</tr>
<tr>
<td>0.00398</td>
<td>0.681</td>
</tr>
<tr>
<td>0.00416</td>
<td>0.810</td>
</tr>
<tr>
<td>0.00435</td>
<td>0.827</td>
</tr>
<tr>
<td>0.00453</td>
<td>0.749</td>
</tr>
<tr>
<td>0.00471</td>
<td>0.581</td>
</tr>
<tr>
<td>0.00489</td>
<td>0.346</td>
</tr>
<tr>
<td>0.00507</td>
<td>0.077</td>
</tr>
<tr>
<td>0.00525</td>
<td>-0.164</td>
</tr>
<tr>
<td>0.00543</td>
<td>-0.320</td>
</tr>
</tbody>
</table>
Motivating example

Tuning fork data

![Graph of tuning fork data](image)

- Pressure on the y-axis.
- Time (sec) on the x-axis.
- The data points form a wave pattern.

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Function = machine
Basic definitions

A function from a set $A$ to a set $B$ is a rule that assigns a unique (single) element $f(x) \in B$ to each element $x \in A$.

- The set $A$ of all possible input values is called the domain of the function.
- The set of all values of $f(x)$ as $x$ varies throughout $A$ is called the range of the function.
### Examples

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain (x)</th>
<th>Range (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = x^2$</td>
<td>$\mathbb{R}$</td>
<td>$[0, \infty)$</td>
</tr>
<tr>
<td>$y = 1/x$</td>
<td>$\mathbb{R} \setminus {0}$</td>
<td>$\mathbb{R} \setminus {0}$</td>
</tr>
<tr>
<td>$y = \sqrt{x}$</td>
<td>$[0, \infty)$</td>
<td>$[0, \infty)$</td>
</tr>
<tr>
<td>$y = \sqrt{4 - x}$</td>
<td>$(-\infty, 4]$</td>
<td>$[0, \infty)$</td>
</tr>
<tr>
<td>$y = \sqrt{1 - x^2}$</td>
<td>$[-1, 1]$</td>
<td>$[0, 1]$</td>
</tr>
</tbody>
</table>

### Exercise

Give the range and domain of the following functions:

- $f(x) = x^2 - 7$
- $g(t) = \frac{1}{\sqrt{t}}$
- $h(x) = \frac{1}{\sqrt{1 - z^2}}$
If \( f \) is a function with domain \( D \), its graph consists of the points in the Cartesian plane whose coordinates are the input-output pairs for \( f \). In set notation, the graph is

\[
\{(x, f(x)) : x \in D\}.
\]

If \((x, y)\) lies on the graph of \( f \), then the value is the height of the graph above the point \( x \) (or below \( x \) if \( f(x) \) is negative).
Evaluating a function from its graph

The graph of a fruit fly population $p$ is shown in the following figure:

(a) Find the populations after 20 and 45 days.
(b) What is the (approximate) range of the population over the time interval $0 \leq t \leq 50$?
Fruit flies
Are these curves graphs of functions?

Not every curve you draw is the graph of a function. A function $f$ can have only one value $f(x)$ for each $x$ in its domain, so no vertical line can intersect the graph of a function more than once.
Linear functions

**Definition**

A function of the form \( f(x) = mx + b \) for constants \( m \) and \( b \), is called a linear function, \( m \) is the slope of the function/line and \( b \) is the intersection with the axis \( y \). Constant functions have slope \( m = 0 \).
Exercise

Give the equation of the linear function that has slope \( m = 2 \) and intersects the \( y \)-axis in 3.

Exercise

Give the equation of the linear function that has slope \( m = 2 \) and goes through the point \((4, 5)\).

Exercise

Give the equation of the linear function that goes through the points \((4, 5)\) and \((-1, 3)\).
A function \( f(x) = x^a \) where \( a \) is a constant, is called a power function.
Power functions

\[ y = x^4 \]

\[ y = x^5 \]

\[ y = x^6 \]

\[ y = x^7 \]

\[ y = x^{-1} \]

\[ y = x^{-2} \]
The functions \( y = x^{1/2} = \sqrt{x} \) and \( y = x^{1/3} = \sqrt[3]{x} \) are the square root and cube root functions, respectively. The domain of the square root function is \([0, \infty)\) but the cube root function is defined for all real \(x\).
A function $p$ is a polynomial if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $n$ is a nonnegative integer and the numbers $a_0, a_1, \ldots, a_n$ are real constants (called the coefficients of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$ then $n$ is called the degree of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called quadratic functions. Likewise, cubic functions are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3.
$y = 8x^4 - 14x^3 - 9x^2 + 11x - 1$

$y = (x - 2)^4(x + 1)^3(x - 1)$
Rational functions

Definition

A rational function is a quotient or ratio of two polynomials:

\[ f(x) = \frac{p(x)}{q(x)} \]

where \( p \) and \( q \) are polynomials.

\[ y = \frac{2x^2 - 3}{7x + 4} \]
Piecewise functions

\[ f(x) = \begin{cases} 
-x, & \text{if } x < 0, \\
x^2, & \text{if } 0 \leq x \leq 1, \\
1, & \text{if } x > 1. 
\end{cases} \]

\[ |x| = \begin{cases} 
-\frac{3}{2}x, & \text{if } x < 0, \\
x, & \text{if } 0 \leq x. 
\end{cases} \]
Other functions

**Definition**
An algebraic function is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots).

**Definition**
Functions of the form $f(x) = a^x$ where the base $a > 0$ is a positive constant and $a \neq 1$, are called exponential functions.

**Definition**
Functions of the form $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant, are called logarithmical functions.
Exponential and logarithmic function

$y = 10^x$
$y = 3^x$
$y = 2^x$

$y = \log_{10} x$
$y = \log_5 x$
$y = \log_2 x$
Trigonometric functions

$$y = \sin x$$

$$y = \cos x$$
Combining Functions Algebraically ($+, −, ·, /$)

Example

Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1 - x}$.

<table>
<thead>
<tr>
<th>Function</th>
<th>Formula</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f + g$</td>
<td>$\sqrt{x} + \sqrt{1 - x}$</td>
<td>[0, 1]</td>
</tr>
<tr>
<td>$f - g$</td>
<td>$\sqrt{x} - \sqrt{1 - x}$</td>
<td>[0, 1]</td>
</tr>
<tr>
<td>$g - f$</td>
<td>$\sqrt{1 - x} - \sqrt{x}$</td>
<td>[0, 1]</td>
</tr>
<tr>
<td>$f \cdot g$</td>
<td>$\sqrt{x} \cdot \sqrt{1 - x} = \sqrt{x(1 - x)}$</td>
<td>[0, 1]</td>
</tr>
<tr>
<td>$f / g$</td>
<td>$\frac{\sqrt{x}}{\sqrt{1 - x}} = \sqrt{\frac{x}{1 - x}}$</td>
<td>[0, 1]</td>
</tr>
<tr>
<td>$g / f$</td>
<td>$\frac{\sqrt{1 - x}}{\sqrt{x}} = \sqrt{\frac{1 - x}{x}}$</td>
<td>(0, 1]</td>
</tr>
</tbody>
</table>
**Composite functions**

**Definition**

If $f$ and $g$ are functions, the composite function $f \circ g$ ("$f$ composed with $g$") is defined by

$$(f \circ g)(x) = f(g(x)).$$

**Exercise**

Give the formulas and domains of the following composite functions

(a) $(f \circ g)(x)$  
(b) $(g \circ f)(x)$  
(c) $(f \circ f)(x)$  
(d) $(g \circ g)(x)$

where $f(x) = \sqrt{x}$ and $g(x) = x + 1$. 
Monotonicity

If the graph of a function climbs or rises as you move from left to right, we say that the function is increasing. If the graph descends or falls as you move from left to right, the function is decreasing.

<table>
<thead>
<tr>
<th>Function</th>
<th>Increasing</th>
<th>Decreasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = x^2 )</td>
<td>([0, \infty))</td>
<td>((-\infty, 0])</td>
</tr>
<tr>
<td>( y = x^3 )</td>
<td>((-\infty, \infty))</td>
<td>Nowhere</td>
</tr>
<tr>
<td>( y = \frac{1}{x} )</td>
<td>Nowhere</td>
<td>((-\infty, 0) \cup (0, \infty))</td>
</tr>
<tr>
<td>( y = \frac{1}{x^2} )</td>
<td>((-\infty, 0))</td>
<td>((0, \infty))</td>
</tr>
<tr>
<td>( y = \sqrt{x} )</td>
<td>([0, \infty])</td>
<td>Nowhere</td>
</tr>
</tbody>
</table>

Definition

A function \( f(x) \) increases (decreases) on an interval \( I \) if \( f(a) \leq f(b) \) \((f(a) \geq f(b))\) for all \( a < b \), where \( a, b \in I \).
Symmetry

symmetry about the $y$-axis

$$y = x^2$$

$(x, y)$ (x, y)

symmetry about the origin

$$y = x^3$$

$(x, y)$  

$(−x, y)$ 

$(−x, y)$
Symmetry

**Definition**

A function $y = f(x)$ is an

- even function of $x$, if $f(-x) = f(x)$,
- odd function of $x$, if $f(-x) = -f(x)$,

for every $x$ in the function’s domain.

**Exercise**

Recognize the symmetry of the functions:

\[ f(x) = x^2, \quad f(x) = x^2 + 1, \quad f(x) = x, \quad f(x) = x + 1. \]
Vertical shifting: $y = f(x) + k$

- Shifts the graph of $f$ up $k$ units if $k > 0$.
- Shifts the graph of $f$ down $|k|$ units if $k < 0$. 

\begin{align*}
  y &= x^2 \\
  y &= x^2 + 3 \\
  y &= x^2 - 1
\end{align*}
Shifting

**Horizontal shifting: $y = f(x + h)$**

- Shifts the graph of $f$ left $h$ units if $h > 0$.
- Shifts the graph of $f$ right $|h|$ units if $h < 0$.

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Mathematics

![Graph showing horizontal shifts](image)
Scaling and reflection

**Vertical scaling:** \( y = c \cdot f(x) \)

- Stretches the graph of \( f \) vertically by a factor of \( c \) if \( c > 1 \).
- Compresses the graph of \( f \) vertically by a factor of \( c \) if \( 0 < c < 1 \).
Scaling and reflection

**Horizontal scaling:** \( y = f(c \cdot x) \)

- Compresses the graph of \( f \) horizontally by a factor of \( c \) if \( c > 1 \).
- Stretches the graph of \( f \) horizontally by a factor of \( c \) if \( 0 < c < 1 \).

\[ y = \sqrt{x} \]
\[ y = \sqrt{3x} \]
\[ y = \sqrt{\frac{1}{2}x} \]
Scaling and reflection

Vertical reflection: \( y = -f(x) \)
Reflects the graph of \( f \) across the \( x \)-axis.

Horizontal reflection: \( y = f(-x) \)
Reflects the graph of \( f \) across the \( y \)-axis.
Main problem: How to draw a graph of a function using transformation steps?

Algorithm

To draw the graph of $y = a \cdot f(bx + c) + d$.

1. Initial step: $y = f(x)$.
2. Horizontal shifting: $y = f(x + c)$.
3. Horizontal scaling and reflection: $y = f(bx + c)$.
4. Vertical scaling and reflection: $y = a \cdot f(bx + c)$.
5. Vertical shifting: $y = a \cdot f(bx + c) + d$. 
Example

\[ f(x) = -\frac{1}{3}(2x - 3)^2 + 2 \]
Draw the graphs of the following functions using transformations:

(a) \( f(x) = 1 - \sqrt{x} \)
(b) \( g(x) = \frac{1}{x-3} \)
(c) \( h(x) = -(x - 2)^2 + 1 \)
(d) \( j(x) = \sqrt[3]{2x - 1} \)
(e) \( k(x) = 2(x + 1)^2 - 1 \)
(f) \( l(x) = (2x + 1)^2 - 1 \)